THE MATHEMATICAL THEORY OF THE MOTION OF ROTATED AND UNROTATED ROCKETS

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CONTENTS

An account is given of the mathematical theory of the motion of a rocket in flight. The aerodynamic forces and couples, and those due to the action of the burning gases, are investigated as fully as possible, and the equations of motion are set up in their most general form. The effects of a variety of disturbing factors, such as wind and asymmetries of design and functioning, are considered. Solutions of the equations, most of which are suitable for numerical computation, are given under various assumptions regarding the form of the axial spin, the aerodynamic lift moment, the acceleration, etc. A thorough investigation of the conditions necessary for stable motion is carried out. The paper concludes with a summary in which the main features of rocket motion, as revealed by the theory, are discussed in general terms.

1. INTRODUCTION

1·1. General description of the problem

The object of the present paper is to provide as comprehensive an account as possible of the mathematical theory of the motion of a rocket in flight. Under the name *rocket* we understand any projectile, rotated or unrotated, which contains a combustion chamber in which some form of propellant is consumed and converted into gases which are rapidly ejected through one or more orifices, and so, by virtue of the law of conservation of momentum, cause the projectile to move. We assume that the propellant is solid in the cold state and that it contains all the materials necessary for complete combustion.† The theory developed can be used to determine the motion of any such projectile provided that it does not possess too great a degree of asymmetry. It is, however, most suited for the investigation of rockets of the kind which have been developed in this country during the 1939–45 war, and the majority of the simpler results obtained have been derived under assumptions which are valid for rockets of this type.

We are chiefly interested in the motion during the period when the propellant is being consumed, i.e. during *burning*. After *all-burnt* the motion does not differ from that of a shell of the same shape, and the ordinary shell ballistic theory is therefore applicable.‡ Also, it is during burning that disturbing factors other than gravity have their greatest effect. In addition, we confine our attention to rockets whose burning time is not too long, i.e. we assume that the angular departure from the initial direction of motion remains small while the propellant is being consumed. This assumption holds for all rockets of the type mentioned above.

The motion of a rocket during its burning period is of quite a different character from that of a shell. This is primarily due to the forward thrust which is exerted by the escaping gases and which is by far the largest force acting on the projectile. Owing to the magnitude of the thrust, it may not be necessary to know the magnitude of the aerodynamic forces and couples (in particular, the drag) to the same degree of accuracy as for a shell, but this advantage is offset in practice by the difficulty of accurately determining the thrust, and by the considerable variation occurring between projectiles.§ In nearly all other respects, however, the problem of predicting the path of a rocket is much more complicated than the similar problem for shell. One of the main reasons for this is the fact that the rocket's motion is very

† Thus projectiles which derive their oxygen from the surrounding atmosphere are excluded.

‡ See, for example, Fowler, Gallup, Lock & Richmond (1920) or Nielsen & Synge (1946).

§ There are several other complications, such as the absence during the burning period of a 'base drag' contribution to the total air resistance.

sensitive to asymmetries of design and functioning, such as are caused by inaccuracies in manufacture, by uneven gas flow and by distortion of the metal components under the severe stresses set up during burning. These asymmetries can produce considerable deviations from the mean trajectory and are the primary cause of the characteristic high dispersion of the rocket as compared with shell fired from guns. Accordingly, it is important to know how the various asymmetries affect the motion, and which of them are of most importance, so that care may be taken to ensure that the relevant manufacturing 'tolerances' may be confined within as narrow limits as possible. A great deal of the complexity of the mathematical theory is due to the fact that perfect axial symmetry is not a legitimate assumption.

The mathematical theory applies to both rotated and unrotated rockets. Unrotated rockets are usually stabilized by fitting fins near the rear end in order to ensure that the lift moment is stabilizing. If the projectile is rotated it may be possible, as for a shell, to stabilize the motion without using fins. However, since the length-diameter ratio of a rocket is usually higher than that of a shell, owing to the necessity of providing space for a combustion chamber, it is not always possible in practice to impart enough spin to stabilize projectiles without fins.

The assumptions made at each stage are clearly stated.

1·2. Historical background and acknowledgements

The first theoretical work on rockets done in this country was begun in 1936 when the Rocket Section of the Ballistics Research Department was formed at Woolwich under the direction of Dr A. D. Crow (now Sir Alwyn Crow). A considerable quantity of work on the theory of motion of rockets and on methods of trajectory calculation was done there by Mr W. R. Cook with the assistance of Mr A. T. Wadley and Dr F. E. Mercer. The two-dimensional theory was developed by them and shaped into a form which proved of great use in the study of the dispersion and deviation of unrotated rockets. The broad essentials of the three-dimensional theory were also laid down on lines based upon the work of Fowler *et al.* (1920). At the beginning of 1938 the further development of the three-dimensional theory was entrusted to Mr C. L. Barham. By the systematic use of complex variables and vector methods he extended the theory and derived solutions of the equations which could be used in numerical computations. His work was embodied in an unpublished report which has for long been known to rocket ballisticians under the affectionate title of 'the tome'. The theory of the deviation of an unrotated rocket due to wind was further extended by him with the help of Messrs E. T. J. Davies and H. G. Haden, and the methods of calculation of the main aerodynamic parameters were also improved by them. In 1942 Professor L. Rosenhead, who was then head of the ballistics section of the Projectile Development Establishment at Aberporth, was anxious that the whole mathematical theory should be reviewed and put upon the most general and satisfactory footing possible; it was at his request and with his encouragement that the author began the systematic revision and extension of the two- and three-dimensional theory giving special attention to the consideration of the motion of rotated rockets, upon which work was then commencing in this country. The results of these investigations were published in 1943 to 1945 in three departmental reports by the author and were later, at the request of the Ministry of Supply,

extended and amplified by him for the purpose of inclusion in a 'monograph' on the exterior ballistics of rockets. This unpublished monograph, upon the mathematical part of which the present paper is based, was written in conjunction with Mr A. T. Wadley and contains, in addition to the mathematical theory, several chapters on its practical applications.

Such, in bare outline, is the history of rocket ballistic theory in Great Britain. The present paper owes much to the labours of those mentioned above and to others; much of the work of the early workers in the field was never written in official report form and has not, accordingly, always received its due recognition.

In the list of references a full list is given of all the published papers to which reference has been made. In his development of the theory the author has made considerable use of the work of Nielsen & Synge and of Kelley & McShane. This work is now openly published, or in process of publication, but appeared originally in reports to the American Office of Scientific Research and Development.

Since the above was written the author has received a copy of a book by J. Barkley Rosser, Robert R. Newton and George L. Gross entitled *Mathematical Theory of Rocket Flight* (New York, 1947). This work contains an account of the theory of motion of a fin-stabilized unrotated rocket;† the results obtained agree with those given in the present paper for such rockets, although the functions $\mathrm{rr}(x)$ and $\mathrm{rj}(x)$ used in the solutions are of a slightly different form from the functions $A(x)$ and $B(x)$ used here.

1·3. Notation

A few preliminary words on the mathematical notation are perhaps required. On glancing at appendices C and D the reader will notice not only that the number of symbols used is very great, but that the notation differs in many respects from that which is customary in shell ballistics. The extreme complication of the subject is sufficient excuse for the multiplicity of the symbols. The differences of usage between rocket and shell ballistics are, on the other hand, attributable to British rocket tradition—which though young is firmly established—and to the essential difference between the two types of projectile which is reflected in the differing forms of the relevant mathematical parameters.

2. THE GENERAL EQUATIONS OF MOTION OF A BODY WHICH IS LOSING MASS

2·1. General

In this section the general equations of motion are derived for any body of variable density which is losing mass from a certain plane portion of its surface.‡ The analysis is first applied to the rocket in § 2·5, where the expressions occurring in the general equations are evaluated as far as possible by the introduction of certain simplifying assumptions. The equations upon which all the later investigations are based are equations (2·5·11, 12). The reader may take these equations for granted if he wishes and proceed to § 3, since the present section is, to a large extent, independent of the rest of the work. For this reason the summary of the main notation is deferred to the beginning of § 3.

† The theory is modified to cover the case where there is a slow axial rotation and gyroscopic effects can be neglected.

‡ Similar results by Gantmacher & Levin (1947) have just come to the author's notice.

2·2. Definitions and notation

Let B be a body which is bounded by a closed surface Σ of invariant shape. All points interior to Σ are regarded as forming part of B. Thus a meaning can be assigned to velocities or positions relative to B.

Let ρ be the mass density at any point P of the body, and let $d\tau$ denote a volume element enclosing P. It is assumed that ρ is not discontinuous at any point, and that the derivative of ρ with respect to the time exists at all points of the body. In the case of the rocket there will be interior surfaces where these conditions are not satisfied, but these surfaces may be regarded as being replaced by layers of arbitrarily small thickness throughout which the conditions are satisfied. The final equations are independent of such discontinuities and will continue to hold in the limiting case.

The body is assumed to be losing mass from a plane portion† S_0 of its surface called the *exit plane*; particles of matter which have left this surface are no longer considered to form part of the body.

The total area of the exit plane S_0 is denoted by Σ_0. S_0 need not consist of a single simply connected part of the surface, but may be composed of several unconnected areas.

The following notation for vectors is adopted. All vectors are printed in clarendon type. Thus $\mathbf{V}$, $\mathbf{OX}$ are vectors of absolute magnitudes V, OX respectively. Here $\mathbf{OX}$ is the line vector from the point O to the point X. A cross denotes a vector product, and a dot a scalar product, e.g.

$$\mathbf{U}\times\mathbf{V}, \quad \mathbf{U}.\mathbf{V}.$$

Let O be a point fixed in space from which the vector $\mathbf{r} = \mathbf{OP}$ is measured. Let G be the centre of gravity of the body at any time t, and let H be any point of B which is fixed relative to B (see figure 1). Write

$$\mathbf{R} = \mathbf{HP}, \quad \mathbf{R}_G = \mathbf{HG}, \quad \mathbf{r}_H = \mathbf{OH}, \quad \mathbf{r}_G = \mathbf{OG}.$$

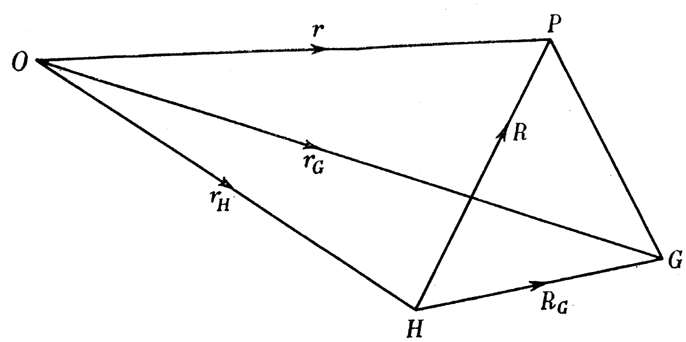

Figure 1

Suppose that $\mathbf{V}_H$ is the velocity of the point H in space. Then if $\mathbf{\Omega}$ is the angular velocity of the body, the velocity $\mathbf{U}$ of any point P fixed in B is

$$\mathbf{U} = \mathbf{V}_H + \mathbf{\Omega}\times\mathbf{R}. \tag{2·2·1}$$

The velocity of any particle of mass at P relative to B is denoted by $\mathbf{v}$.

† The condition that S_0 be plane is not essential but makes for simplicity. In certain cases S_0 may consist of a number of plane but not coplanar areas. The appropriate modifications to the theory are easily made.

The operator d/dt denotes total differentiation with respect to the time t, the motion of the body in space being taken into consideration. The operator $\partial/\partial t$ denotes differentiation with respect to t relative to the body. Thus, if $\mathbf{E}$ is any variable vector,

$$\frac{d}{dt}\mathbf{E} = \frac{\partial}{\partial t}\mathbf{E} + \mathbf{\Omega}\times\mathbf{E}. \tag{2·2·2}$$

Let $\mathbf{V}$ be the velocity of G in space. Then

$$\mathbf{V} = \frac{d}{dt}\mathbf{r}_G = \mathbf{V}_H + \mathbf{\Omega}\times\mathbf{R}_G + \frac{\partial \mathbf{R}_G}{\partial t}. \tag{2·2·3}$$

Here $\partial \mathbf{R}_G/\partial t$ denotes the velocity of G in the body relative to the body.

2·3. The effect of variable density

If m is the total mass, and Q the rate of emission of mass from the exit plane, then

$$m = \int_B \rho\, d\tau \tag{2·3·1}$$

and

$$Q = -\int_B \frac{\partial \rho}{\partial t}\, d\tau = \int_{S_0} \rho v_N\, dS, \tag{2·3·2}$$

where v_N is the component of $\mathbf{v}$ perpendicular to the element dS of the exit plane. Also

$$m\mathbf{R}_G = \int_B \mathbf{R}\rho\, d\tau,$$

and therefore

$$\frac{\partial}{\partial t}(m\mathbf{R}_G) = -Q\mathbf{R}_G + m\frac{\partial \mathbf{R}_G}{\partial t} = \frac{\partial}{\partial t}\int_B \mathbf{R}\rho\, d\tau = \int_B \mathbf{R}\frac{\partial \rho}{\partial t}\, d\tau, \tag{2·3·3}$$

i.e.

$$\frac{\partial \mathbf{R}_G}{\partial t} = \frac{1}{m}\int_B (\mathbf{R}-\mathbf{R}_G)\frac{\partial \rho}{\partial t}\, d\tau. \tag{2·3·4}$$

Now, by considering the motion of the element of mass $\rho\, d\tau$ relative to B in the interval of time $(t, t+\delta t)$, it is seen that

$$\int_B \mathbf{R}\left(\rho + \frac{\partial \rho}{\partial t}\delta t\right) d\tau = \int_B (\mathbf{R}+\mathbf{v}\,\delta t)\,\rho\, d\tau - \delta t \int_{S_0} \mathbf{R}\rho v_N\, dS.$$

Hence

$$\int_B \mathbf{v}\rho\, d\tau = \int_B \mathbf{R}\frac{\partial \rho}{\partial t}\, d\tau + \int_{S_0} \mathbf{R}\rho v_N\, dS \tag{2·3·5}$$

$$= \int_B (\mathbf{R}-\mathbf{R}_G)\frac{\partial \rho}{\partial t}\, d\tau + \int_{S_0} (\mathbf{R}-\mathbf{R}_G)\,\rho v_N\, dS \tag{2·3·6}$$

by (2·3·2). This is the 'continuity equation'.

Define the point N in the exit plane so that

$$Q\mathbf{R}_N = \int_{S_0} \mathbf{R}\rho v_N\, dS, \tag{2·3·7}$$

and write

$$Q\mathbf{W} = \int_{S_0} \mathbf{v}\rho v_N\, dS. \tag{2·3·8}$$

Then N is the *effective centre* of the exit plane, and $\mathbf{W}$ is the effective relative *resultant gas velocity* at this plane.

By equations (2·3·4, 6, 7)

$$\int_B \mathbf{v}\rho\, d\tau = Q(\mathbf{R}_N - \mathbf{R}_G) + m\frac{\partial \mathbf{R}_G}{\partial t}. \tag{2·3·9}$$

2·4. The equations of motion

These are derived from first principles by finding expressions for the linear and angular momentum (about a fixed point) of the whole body, and equating their rates of change to the external forces and couples acting on the system. This method has been chosen in preference to others involving energy considerations in the hope of avoiding the many pitfalls which abound in this subject.

The linear momentum is, by (2·2·1, 3) and (2·3·9),

$$\mathbf{P} = \int_B (\mathbf{U}+\mathbf{v})\,\rho\,d\tau = m\mathbf{V}_H + \mathbf{\Omega}\times\int_B \mathbf{R}\rho\,d\tau + \int_B \mathbf{v}\rho\,d\tau$$

$$= m\mathbf{V}_H + m\mathbf{\Omega}\times\mathbf{R}_G + \int_B \mathbf{v}\rho\,d\tau = m\mathbf{V} + Q(\mathbf{R}_N - \mathbf{R}_G). \qquad (2·4·1)$$

The angular momentum about O is

$$\mathbf{h} = \int_B \mathbf{r}\times(\mathbf{U}+\mathbf{v})\,\rho\,d\tau = \mathbf{h}_G + \mathbf{r}_G\times\mathbf{P} + \int_B (\mathbf{R}-\mathbf{R}_G)\times\mathbf{v}\rho\,d\tau, \qquad (2·4·2)$$

where

$$\mathbf{h}_G = \int_B (\mathbf{R}-\mathbf{R}_G)\times\mathbf{U}\rho\,d\tau = \int_B (\mathbf{R}-\mathbf{R}_G)\times(\mathbf{\Omega}\times\mathbf{R})\,\rho\,d\tau \qquad (2·4·3)$$

by (2·2·1) and the definition of $\mathbf{R}_G$.

Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be unit vectors in the directions of the principal axes of inertia through G, and let A, B and C be the corresponding moments. $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are taken to be a right-handed set. Let $\boldsymbol{\omega}_C$ be their angular velocity relative to the body. Expressions for $\boldsymbol{\omega}_C$ are given in § 2·6.

Write

$$\Omega_a = \mathbf{a}.\mathbf{\Omega}, \quad \Omega_b = \mathbf{b}.\mathbf{\Omega}, \quad \Omega_c = \mathbf{c}.\mathbf{\Omega}. \qquad (2·4·4)$$

Then

$$\mathbf{h}_G = \int_B (\mathbf{R}-\mathbf{R}_G)\times(\mathbf{\Omega}\times\mathbf{R})\,\rho\,d\tau = \mathbf{a}A\Omega_a + \mathbf{b}B\Omega_b + \mathbf{c}C\Omega_c, \qquad (2·4·5)$$

and

$$\frac{d}{dt}\mathbf{h}_G = \mathbf{a}\frac{d}{dt}(A\Omega_a) + \mathbf{b}\frac{d}{dt}(B\Omega_b) + \mathbf{c}\frac{d}{dt}(C\Omega_c) + (\mathbf{\Omega}+\boldsymbol{\omega}_C)\times\mathbf{h}_G. \qquad (2·4·6)$$

Let $\mathbf{L}$ denote the sum of all the external forces acting on the body, and let $\mathbf{M}$ denote the total couple about the centre of gravity. In the case of the rocket, $\mathbf{L}$ and $\mathbf{M}$ include all aerodynamic forces and couples, gravity, and differences in gas and air pressure at the exit plane; also any reactional forces acting on the projectile while on the projector. The total couple about the fixed point O is

$$\mathbf{M} + \mathbf{r}_G\times\mathbf{L}.$$

Consider now the linear and angular momentum lost in the interval of time $(t, t+\delta t)$. The momentum at the time $t+\delta t$ of B and of the mass emitted during the interval is

$$\mathbf{P} + \delta\mathbf{P} + \delta t\int_{S_0} \rho v_N(\mathbf{v}+\mathbf{V}_H+\mathbf{\Omega}\times\mathbf{R})\,dS.$$

Hence

$$\delta\mathbf{P} + \delta t\int_{S_0} \rho v_N(\mathbf{v}+\mathbf{V}_H+\mathbf{\Omega}\times\mathbf{R})\,dS = \mathbf{L}\delta t;$$

i.e. by (2·3·7, 8) and (2·4·1),

$$\frac{d}{dt}\mathbf{P} = \frac{d}{dt}\{m\mathbf{V} + Q(\mathbf{R}_N - \mathbf{R}_G)\} = \mathbf{L} - Q(\mathbf{W} + \mathbf{V}_H + \mathbf{\Omega}\times\mathbf{R}_N). \qquad (2·4·7)$$

This may be written in the form

$$m\frac{d}{dt}\mathbf{V} = \mathbf{L}-\frac{dQ}{dt}(\mathbf{R}_N-\mathbf{R}_G)-Q\left\{\mathbf{W}+2\mathbf{\Omega}\times(\mathbf{R}_N-\mathbf{R}_G)+\frac{\partial\mathbf{R}_N}{\partial t}-2\frac{\partial\mathbf{R}_G}{\partial t}\right\}. \qquad (2\cdot4\cdot8)$$

Now consider the angular momentum after the interval δt. It is, by (2·4·2),

$$\mathbf{h}_G+\delta\mathbf{h}_G+(\mathbf{r}_G+\delta\mathbf{r}_G)\times(\mathbf{P}+\delta\mathbf{P})+\int_B(\mathbf{R}-\mathbf{R}_G-\delta\mathbf{R}_G)\times(\mathbf{v}+\delta\mathbf{v})\,(\rho+\delta\rho)\,d\tau$$
$$+\delta t\int_{S_0}\rho v_N\mathbf{r}\times(\mathbf{v}+\mathbf{V}_H+\mathbf{\Omega}\times\mathbf{R})\,dS.$$

Hence

$$\frac{d}{dt}\mathbf{h}_G+\frac{d}{dt}\left\{\mathbf{r}_G\times\mathbf{P}+\int_B(\mathbf{R}-\mathbf{R}_G)\times\mathbf{v}\rho\,d\tau\right\}+\int_{S_0}\rho v_N\mathbf{r}\times(\mathbf{v}+\mathbf{V}_H+\mathbf{\Omega}\times\mathbf{R})\,dS = \mathbf{M}+\mathbf{r}_G\times\mathbf{L}.$$

Therefore, by (2·3·2, 7, 8), (2·4·1, 7) and (2·2·3),

$$\frac{d}{dt}\mathbf{h}_G = \mathbf{M}+\mathbf{P}\times\frac{d}{dt}\mathbf{r}_G-\frac{d}{dt}\left\{\int_B(\mathbf{R}-\mathbf{R}_G)\times\mathbf{v}\rho\,d\tau\right\}-\int_{S_0}\rho v_N(\mathbf{R}-\mathbf{R}_G)\times(\mathbf{v}+\mathbf{V}_H+\mathbf{\Omega}\times\mathbf{R})\,dS$$
$$= \mathbf{M}-\int_{S_0}\rho v_N(\mathbf{R}-\mathbf{R}_G)\times\{\mathbf{v}+\mathbf{\Omega}\times(\mathbf{R}-\mathbf{R}_G)\}\,dS$$
$$-\frac{d}{dt}\left\{\int_B(\mathbf{R}-\mathbf{R}_G)\times\mathbf{v}\rho\,d\tau\right\}+Q(\mathbf{R}_N-\mathbf{R}_G)\times\frac{\partial\mathbf{R}_G}{\partial t}. \qquad (2\cdot4\cdot9)$$

Now define the point M on the exit plane, and the distance q_2, by

$$\int_B\mathbf{R}\times\mathbf{v}\rho\,d\tau = \mathbf{R}_M\times\int_B\mathbf{v}\rho\,d\tau+q_2\int_B\mathbf{v}\rho\,d\tau. \qquad (2\cdot4\cdot10)$$

In the application to the rocket, M may be regarded as the point in the exit plane where the 'thrust' acts. M need not coincide with N. With this notation, and by (2·3·9), (2·4·9) becomes

$$\frac{d}{dt}\mathbf{h}_G = \mathbf{M}-\int_{S_0}\rho v_N(\mathbf{R}-\mathbf{R}_G)\times\{\mathbf{v}+\mathbf{\Omega}\times(\mathbf{R}-\mathbf{R}_G)\}\,dS-(\mathbf{R}_M-\mathbf{R}_G)\times\frac{\partial}{\partial t}\int_B\mathbf{v}\rho\,d\tau$$
$$-\frac{\partial\mathbf{R}_M}{\partial t}\times\int_B\mathbf{v}\rho\,d\tau-\mathbf{\Omega}\times\left\{(\mathbf{R}_M-\mathbf{R}_G)\times\int_B\mathbf{v}\rho\,d\tau\right\}-\frac{d}{dt}\left\{q_2\int_B\mathbf{v}\rho\,d\tau\right\}. \qquad (2\cdot4\cdot11)$$

Equations (2·4·8, 11) are the general equations of motion. The left-hand side of (2·4·11) is defined by (2·4·5, 6).

2·5. Simplification of the equations

The equations which have been obtained are now applied specifically to the rocket, and several reasonable simplifying assumptions are made. It is assumed that the positions of the points M and N are fixed in B. Then

$$\frac{\partial\mathbf{R}_M}{\partial t} = \frac{\partial\mathbf{R}_N}{\partial t} = 0. \qquad (2\cdot5\cdot1)$$

The body consists of two parts, B_1 which remains of constant density throughout, and B_2 which consists of those parts which change density at some stage during the motion, i.e. B_2 consists of the charge and interior gases. Let G_1 be the centre of gravity of B_1. It is assumed that G_1 remains fixed in the body. Then the velocity of the centre of gravity inside the body is given by

$$\frac{\partial \mathbf{R}_G}{\partial t} = \frac{Q}{m}(\mathbf{R}_G - \mathbf{R}_{G_1}) = \frac{Q}{m}\mathbf{G}_1\mathbf{G}. \tag{2·5·2}$$

For it follows from (2·3·4), on taking $H = G_1$, that

$$m\frac{\partial \mathbf{R}_G}{\partial t} - Q\mathbf{R}_G = \frac{\partial}{\partial t}\left\{\int_{B_1} \mathbf{R}\rho\, d\tau + \int_{B_2} \mathbf{R}\rho\, d\tau\right\} = 0 + \frac{\partial}{\partial t}\left\{\mathbf{R}_{G_1}\int_{B_2} \rho\, d\tau\right\}$$

$$= \frac{\partial}{\partial t}\left\{m\mathbf{R}_{G_1} - \mathbf{R}_{G_1}\int_{B_1} \rho\, d\tau\right\} = -Q\mathbf{R}_{G_1}.$$

It is clear that (2·5·2) does not depend upon the particular choice of the fixed point H made in the above proof. It is convenient from now on to take H at the instantaneous position of G at the time t (see figure 2). Let q_1 be the distance of G_1 from N, and write

$$\mathbf{NG}_1 = q_1\mathbf{n}, \tag{2·5·3}$$

where $\mathbf{n}$ is of unit magnitude and is fixed relative to the body. Then, by (2·3·9) and (2·5·2, 3), since $\mathbf{R}_G = 0$,

$$\int_B \mathbf{v}\rho\, d\tau = Q\mathbf{R}_N + m\frac{\partial \mathbf{R}_G}{\partial t} = -Qq_1\mathbf{n}, \tag{2·5·4}$$

and

$$\frac{\partial}{\partial t}\int_B \mathbf{v}\rho\, d\tau = -\frac{dQ}{dt}q_1\mathbf{n}, \quad \frac{\partial}{\partial t}\left\{q_2\int_B \mathbf{v}\rho\, d\tau\right\} = -\mathbf{n}q_1\frac{d}{dt}(Qq_2). \tag{2·5·5}$$

Little is known of the magnitude of q_2, but it is probable that it is small and that it remains approximately constant (see § 3·61).

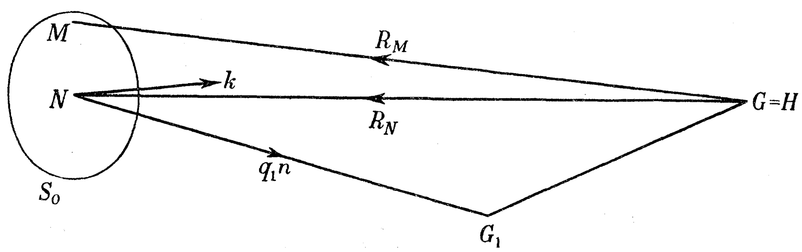

FIGURE 2

Now consider the integral

$$\int_{S_0} \rho v_N \mathbf{R}\times(\mathbf{\Omega}\times\mathbf{R})\, dS.$$

Let $\mathbf{u}$ be a vector from N to any point of the exit plane, so that, by the definition of N,

$$\int_{S_0} \rho v_n \mathbf{u}\, dS = 0.$$

Since $\mathbf{R}\times(\mathbf{\Omega}\times\mathbf{R}) = \mathbf{R}_N\times(\mathbf{\Omega}\times\mathbf{R}_N) + \mathbf{u}\times(\mathbf{\Omega}\times\mathbf{R}_N) + \mathbf{R}_N\times(\mathbf{\Omega}\times\mathbf{u}) + \mathbf{u}\times(\mathbf{\Omega}\times\mathbf{u})$,

it follows that

$$\int_{S_0} \rho v_N \mathbf{R}\times(\mathbf{\Omega}\times\mathbf{R})\, dS = Q\mathbf{R}_N\times(\mathbf{\Omega}\times\mathbf{R}_N) + \int_{S_0} \rho v_N \mathbf{u}\times(\mathbf{\Omega}\times\mathbf{u})\, dS.$$

In order to evaluate the integral on the right-hand side it is necessary to know in what manner the gas velocity and density vary across the exit plane S_0. It is believed that in practice the variations of ρv_N across S_0 can be neglected, and accordingly it is assumed that ρv_N takes the same value at all points of S_0. Then N is the geometrical centre of S_0, and it is assumed that S_0 is such that

$$\int_{S_0} u_1 u_2 \, dS = 0,$$

where u_1, u_2 are the components of $\mathbf{u}$ along any two perpendicular directions in S_0. This is the case if, for example, S_0 has symmetry of order greater than 1 about N (see § 3·4).

Let $\mathbf{k}$ be a unit vector through N normal to S_0 towards the interior of B, and let k_e be the radius of gyration† of S_0 about $\mathbf{k}$. Write

$$NG = l. \tag{2·5·6}$$

Then
$$\int_{S_0} \rho v_N \mathbf{u} \times (\boldsymbol{\Omega} \times \mathbf{u}) \, dS = \tfrac{1}{2} Q k_e^2 (\boldsymbol{\Omega} + \mathbf{k}\Omega_k). \tag{2·5·7}$$

Hence
$$\int_{S_0} \rho v_N \mathbf{R} \times (\boldsymbol{\Omega} \times \mathbf{R}) \, dS = Q\mathbf{R}_N \times (\boldsymbol{\Omega} \times \mathbf{R}_N) + \tfrac{1}{2} Q k_e^2 (\boldsymbol{\Omega} + \mathbf{k}\Omega_k). \tag{2·5·8}$$

Consider now the integral
$$\int_{S_0} \rho v_N \mathbf{R} \times \mathbf{v} \, dS.$$

On the exit plane, let
$$\mathbf{v} = \mathbf{W} + \mathbf{v}_1,$$

so that
$$\int_{S_0} \rho v_N \mathbf{R} \times \mathbf{v} \, dS = Q\mathbf{R}_N \times \mathbf{W} - \mathbf{G}_R + \mathbf{n} q_1 \frac{d}{dt}(Q q_2), \tag{2·5·9}$$

where
$$\mathbf{G}_R = -\int_{S_0} \rho v_N \mathbf{R} \times \mathbf{v}_1 \, dS + \mathbf{n} q_1 \frac{d}{dt}(Q q_2). \tag{2·5·10}$$

$\mathbf{G}_R$ may be regarded as the rotational couple due to the jet, since it is only appreciable when S_0 is the exit plane of a multiple offset nozzle system. The second term on the right-hand side of (2·5·10) is small in comparison with the first term which provides the main couple.

We can now state the equations of motion in a simplified form. They are, by (2·4·8, 11) and (2·5·4, 5, 8, 9),

$$m\frac{d}{dt}\mathbf{V} = \mathbf{L} - Q\left\{\mathbf{W} + 2\boldsymbol{\Omega} \times \mathbf{R}_N + \frac{2Q}{m}(\mathbf{R}_N + q_1 \mathbf{n})\right\} - \frac{dQ}{dt}\mathbf{R}_N \tag{2·5·11}$$

and
$$\begin{aligned} \frac{d}{dt}\mathbf{h}_G &= \mathbf{a}\frac{d}{dt}(A\Omega_a) + \mathbf{b}\frac{d}{dt}(B\Omega_b) + \mathbf{c}\frac{d}{dt}(C\Omega_c) + (\boldsymbol{\Omega} + \boldsymbol{\omega}_C) \times (\mathbf{a}A\Omega_a + \mathbf{b}B\Omega_b + \mathbf{c}C\Omega_c) \\ &= \mathbf{M} + \mathbf{G}_R - Q\mathbf{R}_N \times (\mathbf{W} + \boldsymbol{\Omega} \times \mathbf{R}_N) - \tfrac{1}{2} Q k_e^2 (\boldsymbol{\Omega} + \mathbf{k}\Omega_k) \\ &\quad + \frac{dQ}{dt} q_1 \mathbf{R}_M \times \mathbf{n} + Q q_1 \boldsymbol{\Omega} \times (\mathbf{R}_M \times \mathbf{n}) + Q q_1 q_2 \boldsymbol{\Omega} \times \mathbf{n}. \end{aligned} \tag{2·5·12}$$

2·6. The motion of the principal axes of inertia inside the body

It can be shown that, for a general body,

$$\boldsymbol{\omega}_C = -\int_B \left\{\frac{YZ\mathbf{a}}{B-C} + \frac{ZX\mathbf{b}}{C-A} + \frac{XY\mathbf{c}}{A-B}\right\} \frac{\partial \rho}{\partial t} \, d\tau, \tag{2·6·1}$$

† If the rocket has a single nozzle, S_0 will usually be a circle of radius b_1 and $k_e^2 = \frac{1}{2} b_1^2$; an expression for k_e^2 when a multiple nozzle is used is given in § 9. Formula (2·5·7) continues to hold without the assumption of constant ρv_n if k_e and $\mathbf{k}$ are defined suitably.

where $$X = \mathbf{R}.\mathbf{a}, \quad Y = \mathbf{R}.\mathbf{b}, \quad Z = \mathbf{R}.\mathbf{c}, \tag{2·6·2}$$

and $\mathbf{R}$ is measured from G. Under certain assumptions regarding the symmetry and disposition of the charge (2·6·1) may be evaluated approximately giving

$$\boldsymbol{\omega}_C = Q\left\{\frac{Y_1 Z_1 \mathbf{a}}{B-C} + \frac{Z_1 X_1 \mathbf{b}}{C-A} + \frac{X_1 Y_1 \mathbf{c}}{A-B}\right\}, \tag{2·6·3}$$

where $$\mathbf{GG}_1 = \mathbf{R}_N + q_1 \mathbf{n} = X_1 \mathbf{a} + Y_1 \mathbf{b} + Z_1 \mathbf{c}.$$

So far nothing has been assumed about the relative magnitudes of the moments of inertia A, B and C. In all cases of practical application, however, the rocket's shape and mass distribution is such that the transverse moments A and B are approximately equal, and are both considerably greater than the longitudinal moment C. When $A = B$, the precise directions of the vectors $\mathbf{a}$ and $\mathbf{b}$ are not uniquely determined, and any perpendicular directions in the plane normal to $\mathbf{c}$ may be chosen. In this case it is $\partial\mathbf{c}/\partial t$ and not $\boldsymbol{\omega}_C$ which is of interest, and it can be shown, from (2·6·1), that

$$\frac{\partial \mathbf{c}}{\partial t} = \boldsymbol{\omega}_C \times \mathbf{c} = \frac{1}{A-C}\int_B \{\mathbf{R} - (\mathbf{R}.\mathbf{c})\,\mathbf{c}\}\,(\mathbf{R}.\mathbf{c})\,\frac{\partial \rho}{\partial t}\,d\tau. \tag{2·6·4}$$

3. DESCRIPTION OF NOTATION AND EXTERNAL FORCE SYSTEM

3·1. General

In this section a general description is given of the main notation, assumptions, and definitions employed. The external forces acting on the rocket are derived and expressed in forms which display their dependence upon the various parameters, such as velocity and axial spin.

3·2. Description of the main notation

The main notation used in the succeeding sections is defined here. It differs slightly from that employed in § 2. All other symbols are defined as they arise. A full index of the symbols employed in the work is to be found in appendix D.

3·21. *Co-ordinate system*

The various systems of co-ordinate axes used, together with other vectors descriptive of the rocket and its motion, are shown in figure 3. In this figure all lettered points, except O, lie on a sphere of unit radius whose centre O lies at the centre of gravity of the rocket. All lines in the figure are arcs of great circles. Thus if P and Q are points on the sphere, OP and OQ are unit vectors, and PQ is an arc of magnitude equal to the angle POQ.

3·211. *Vectors whose directions are fixed in space*

OV is the upward vertical.

OX, **OY**, **OZ** are a right-handed set of mutually perpendicular axes whose directions are fixed in space, and from which the Eulerian angles θ, ϕ and ψ are measured.

OZ is the direction of projection (see § 3·7).

OX lies in the vertical plane VOZ through OZ and in the downward direction so that the angle VOX is obtuse.

$VZ = \frac{1}{2}\pi - \alpha$, where α is the quadrant elevation (Q.E.).

3·212. *The motion of the centre of gravity*

OT is the direction of motion of O in space at any instant. This includes the motion of O inside the rocket. Thus **OT** is the *tangent to the trajectory*.

$ZT = \Theta$ (the *angular deviation* of the trajectory).

$\angle XZT = \Psi$.

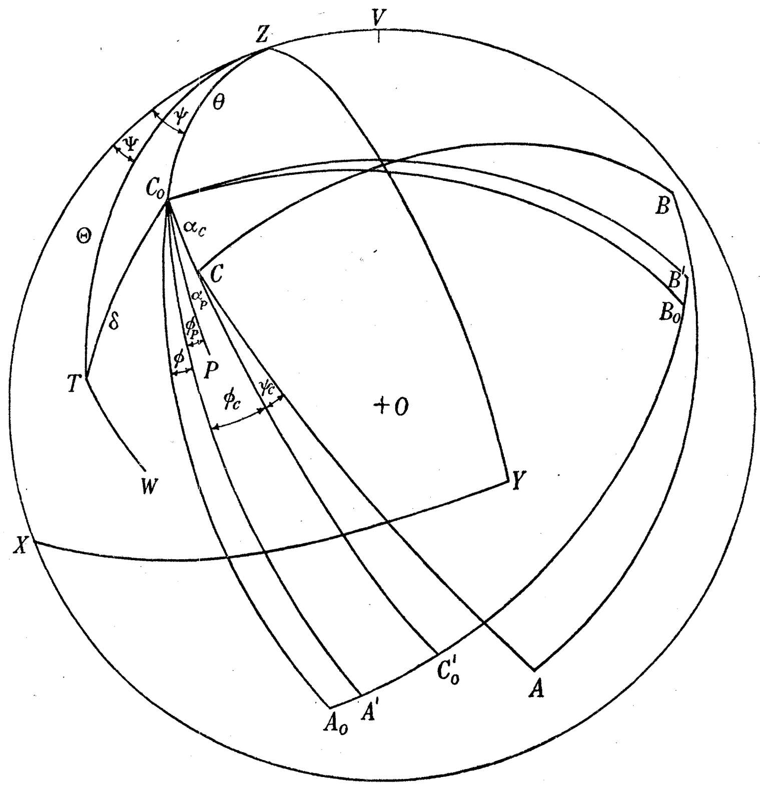

FIGURE 3

3·213. *The motion of the rocket's axis*

$\mathbf{OC_0}$ is the direction of the rocket's axis. This direction is assumed to be fixed in relation to the body of the rocket. The point O will, of course, move relative to the rocket as the propellant is consumed. It is convenient to define $\mathbf{OC_0}$ with regard to the aerodynamic forces (see definition 2, § 3·5).

$\mathbf{OA_0}$, $\mathbf{OB_0}$, $\mathbf{OC_0}$ are a right-handed set of mutually perpendicular axes, not fixed in the body, with Eulerian angles θ, 0, ψ measured from **OX, OY, OZ.**

$\mathbf{OA_0}$ lies in the plane ZOC_0 so that the angle ZOA_0 is obtuse.

$ZC_0 = \theta$, $\angle XZC_0 = \psi$.

$TC_0 = \delta$ (the *yaw*).

The angle between the planes TOC_0 and XOZ is denoted by χ.

OS is the normal to the plane of yaw C_0OT. S lies on the arc A_0B_0. **OT, OS, OS′** are a right-handed set of mutually perpendicular axes.

3·214. *Vectors descriptive of certain asymmetries of design and functioning*

In order to take into consideration the effect of the rotation of the rocket it is necessary to choose a *reference plane* C_0OA' in the rocket which passes through $\mathbf{OC_0}$ and whose normal

OB′ is fixed in direction relative to the rocket's body. The particular plane selected is immaterial, and will be chosen in a convenient manner in § 3·3.

OA′, **OB′**, $\mathbf{OC}_0$ are a right-handed mutually perpendicular set of axes with Eulerian angles θ, ϕ, ψ relative to **OX**, **OY**, **OZ**. The point A' lies on A_0B_0, and $\angle A'C_0A_0 = \phi$.

Axes of inertia. Let **OA**, **OB**, **OC** be the principal axes of inertia, chosen so that **OC** lies close to $\mathbf{OC}_0$. Let the arc C_0C produced cut A_0B_0 in C_0'.

$$C_0C = \alpha_C, \quad \angle ACC_0' = \psi_C.$$

If the moments about **OA** and **OB** remain exactly equal throughout the motion, the precise directions of **OA** and **OB** are indeterminate and may be chosen in any definite manner, e.g. to make $\psi_C = 0$.

Asymmetric gas flow. The directions of the vectors **OR**, **ON′**, **OM′**, **OK**, **OG′** and **OL** are now described. These vectors all lie along OC_0 for a rocket which is perfectly symmetrical about $\mathbf{OC}_0$ both as regards form and the combustion of the propellant (see note on symmetry in § 3·4). For a normal slightly asymmetrical projectile they are inclined at small angles to $\mathbf{OC}_0$.

OR is the direction of the *thrust*, i.e. it is in the opposite direction to the resultant gas velocity **W** at the exit plane. This direction is defined by (2·3·8). The angle $\alpha_R = \angle ROC_0$ is usually referred to as the *jet malalinement.*

N′O defines the direction of the *centre of the exit plane*, i.e. the line joining the centre N (defined by (2·3·7)) to the centre of gravity O cuts the unit sphere in N', when produced.

In the same way **M′O** defines the position of M, the *point of action of the thrust*, i.e. the line joining the point M on the exit plane (defined by (2·4·10)) to the centre of gravity O cuts the unit sphere in M', when produced.

OK is the direction of the *normal to the exit plane* towards the interior. It is the direction of the unit vector **k** defined in § 2·5.

OG′ is the direction of the line NG_1 joining N, the centre of the exit plane, to G_1, the centre of gravity of the charge and burning gases in the rocket, i.e. of the vector **n** of § 2·5.

OL is the axis about which the *rotational couple* $\mathbf{G}_R$ (defined in (2·5·10)) acts.

The angles determining the positions of the seven points C, R, N', M', K, G' and L are defined in the following table. The letter P denotes any one of these seven points. The angles associated with such a point P are shown in figure 3.

point P	C_0P	$\angle PC_0A_0$	reference
C	α_C	$\phi+\phi_C$	axis of inertia
R	α_R	$\phi+\phi_R$	thrust
N'	α_N	$\phi+\phi_N$	exit-plane centre
M'	α_M	$\phi+\phi_M$	thrust-application point
K	α_K	$\phi+\phi_K$	exit-plane normal
G	α_G	$\phi+\phi_G$	charge centre of gravity
L	α_L	$\phi+\phi_L$	rotational torque axis

In general, all the angles mentioned in the table will vary during burning.

In the case of a perfectly symmetric projectile all seven angles α_P vanish. It is not, of course, possible in practice to achieve perfect symmetry, but, by working to certain specified 'tolerances', the manufacturers can keep the various malalinements and asymmetries within fairly small margins. These tolerances are applied to the alinement, concentricity, etc., of the charge and the metal components, so that their connexion with the asymmetries described above is only indirect. Nevertheless it is convenient and customary to refer to the fourteen angles α_p and ϕ_p as tolerances, or tolerance angles, since it is believed that they arise mainly from faults in manufacture and assembly.

3·215. *Wind*

OW is the direction in which the wind is blowing in the neighbourhood of the rocket.

$$WZ = \xi, \quad \angle WZX = \xi_1, \quad WC_0 = \xi_2.$$

If the wind speed is w, its velocity, in vector form, is $\mathbf{w} = w\mathbf{OW}$.

3·22. *Inertial and other characteristics of the rocket*

The masses, moments of inertia, and other relevant dimensions and parameters are defined here.

3·221. *Complete weapon*

This consists of all particles of mass interior to the external surface and the exit plane:

m is the total mass of the rocket at any instant.

l is the distance of the centre of gravity from N the centre of the exit plane.

$2a$ is the maximum calibre of the projectile excluding fins.

$2a_M$ is the maximum overall diameter of the projectile including fins, if any.

A, B and C are the principal moments of inertia about **OA**, **OB** and **OC**, respectively.

3·222. *The propellant*

This is assumed to consist of a number of sticks of solid charge. The moments of inertia of the charge, at any instant, depend upon the shape, number and disposition of the individual sticks, as well as upon the rate of burning on the charge surfaces. Formulae for some simple designs of charge are given in § 9·2.

q is the distance of the centre of gravity of the complete weapon from the centre of gravity G_1 of the charge, before burning commences.

3·223. *The exit plane of the nozzle system*

There may be one or more nozzles. Only the main features of the nozzle system are mentioned here. Some special designs are discussed in § 9.

The exit plane consists of all the cross-sections of the nozzles at their rear ends. These cross-sections are assumed to be coplanar; the modifications necessary when this is not the case can be made without difficulty.

$q_1 = l - q$ is the distance between G_1 and the centre N of the exit plane.

q_2 is defined by (2·4·10). See § 3·61.

Σ_0 is the area of the exit plane.

k_e is the radius of gyration of the exit plane about the normal to it through N.

W is the resultant efflux velocity of the gases relative to the rocket on passing the exit plane; it is defined by (2·3·8).

Q is the rate of emission of gas from the exit plane.

3·23. *The motion of and about the centre of gravity*

t is the time in seconds measured from the instant of ignition, i.e. the instant when the rocket first begins to move on the projector.

V is the velocity of the centre of gravity of the rocket in space; it includes the motion in the projectile due to the alteration in mass distribution caused by the consumption of the propellant; $\mathbf{V} = V\mathbf{OT}$.

$f = dV/dt$ is the acceleration of the centre of gravity along the trajectory.

s is the arc length of the trajectory, i.e. the distance traversed by the centre of gravity since the instant $t = 0$.

$\mathbf{h}_G$ is defined by (2·4·3) and is the angular momentum of the rocket about O. In forming this angular momentum, the velocity of the interior gases relative to the rocket is ignored. Thus, to a high degree of accuracy, $\mathbf{h}_G$ is the angular momentum of the solid components alone.

$\boldsymbol{\omega}_C$ is the angular velocity of the principal axes of inertia of the rocket relative to the rocket. A general expression for $\boldsymbol{\omega}_C$ is given in (2·6·1).

$\boldsymbol{\Omega}$ is the angular velocity of the rocket about its centre of gravity.

r is the spin of the rocket about its axis, i.e. $r = \boldsymbol{\Omega} \cdot \mathbf{OC}_0$.

$\boldsymbol{\omega}$ is the angular velocity of the axis about O, i.e. $\boldsymbol{\omega} = \boldsymbol{\Omega} - r\mathbf{OC}_0$.

σ is the total angle turned through by the projectile about its axis $\mathbf{OC}_0$.

3·24. *Suffixes*

For the symbols

$$A, B, C, f, l, m, r, s, t, V, \quad \delta, \mathrm{Z}, \zeta, \mathrm{H}, \eta, \Theta, \theta, \Lambda, \Xi, \sigma,$$

a double zero suffix (e.g. m_{00}) refers to the instant of ignition when the motion first commences.

A single zero suffix (e.g. m_0) refers to the instant of launch when the projectile ceases all contact with the projector.

A unit suffix (e.g. m_1) refers to the instant $t = t_1$, when the propellant has been completely consumed.

Owing to the multitude of symbols required and the limited number of roman and greek letters available, it has been found necessary to apply numerical suffixes to symbols to denote quantities which are not specifically associated with these three instants. No confusion should arise, however, since all such symbols are defined clearly as they arise, and no symbol possesses more than one meaning in the same section. Literal suffixes are also employed. For further information see the index of symbols (appendix D).

3·25. *Differentiation*

Differentiation with respect to the time t is denoted by a dot, e.g. $\dot{\theta}$; differentiation with respect to the distance travelled s by a dash, e.g. $P'(s)$.

3·3. Complex notation

The main object of the theory is to solve the equations of motion and obtain mathematical expressions for the angular deviation of the trajectory, the yaw and the rate of turn of the axis, together with their orientations about the direction of projection $\mathbf{OZ}$ and the tangent of the trajectory **OT**. When these quantities, together with the spin and velocity, are known throughout burning, the motions of and about the centre of gravity are completely determined. In addition, their values at burnt provide the 'initial conditions' from which the subsequent motion of the projectile may be calculated as for a shell.

The quantities which arise most naturally in the mathematics are the complex quantities:

$$\text{Complex angular deviation:} \quad Z = \Theta\, e^{i\Psi}, \tag{3·3·1}$$

$$\text{Complex inclination of axis:} \quad \zeta = \theta\, e^{i\psi}, \tag{3·3·2}$$

$$\text{Complex yaw:} \quad \Xi = \zeta - Z = \delta\, e^{i\chi}. \tag{3·3·3}$$

The absolute values Θ, θ and δ are the magnitude of the angular deviation, inclination of the axis, and yaw. The angles Ψ, ψ and χ define the orientations relative to the direction of projection (Ψ, ψ) or to the direction of motion (χ) (see figure 3).

The total angular velocity $\mathbf{\Omega}$ of the rocket is, from § 3·213 and figure 3,

$$\mathbf{\Omega} = \dot{\psi}\,\mathbf{OZ} + \dot{\phi}\,\mathbf{OC}_0 + \dot{\theta}\,\mathbf{OB}_0, \tag{3·3·4}$$

so that

$$r = \mathbf{\Omega}\,.\,\mathbf{OC}_0 = \dot{\phi} + \dot{\psi}\cos\theta, \tag{3·3·5}$$

and

$$\boldsymbol{\omega} = \mathbf{\Omega} - r\mathbf{OC}_0 = -\dot{\psi}\sin\theta\,\mathbf{OA}_0 + \dot{\theta}\,\mathbf{OB}_0. \tag{3·3·6}$$

We now introduce the following important assumption:

A 1. *During the burning period of the rocket, the following quantities are small:*

(*a*) $\Theta, \theta, \delta;\ \alpha_C, \alpha_G, \alpha_K, \alpha_L, \alpha_M, \alpha_N, \alpha_R;\ w/V;\ a_M r/V;\ k_e^2/l^2,$

(*b*) $\dot{\theta}, \boldsymbol{\omega};\ d\Theta/dt;\ d\alpha_C/dt.$

The quantities listed are called *first order* quantities. Those in group (*a*) are pure numbers, while those in group (*b*) are angular velocities.

When we say an angular velocity such as $\dot{\theta}$, $\boldsymbol{\omega}$ or $d\alpha_C/dt$ is small, we mean that the ratio of the linear velocity of any point of the body due to the angular velocity about the centre of gravity is small in comparison with the total velocity V of the centre of gravity; i.e. if $\mathfrak{d}$ is the distance of the furthest point of the rocket from the centre of gravity, then $\mathfrak{d}\,\dot{\theta}\,V$, $\mathfrak{d}\omega/V$ and $\mathfrak{d}(d\alpha_C/dt)\,V$ are small. For the angular velocity $d\Theta/dt$, which is not associated with the motion of the projectile about its centre of gravity, a different definition or smallness is required. It is convenient in this case to suppose that $d\Theta/dt$ is of the order of g/V.

It follows from the assumption that the cosines of the angles listed may be replaced by unity, and the sines by the angles themselves. Accordingly we have

$$r = \dot{\phi} + \dot{\psi}, \tag{3·3·7}$$

and

$$\omega = \sqrt{(\dot{\theta}^2 + \dot{\psi}^2\theta^2)}. \tag{3·3·8}$$

It is convenient to choose the reference plane in the body at this juncture (see § 3·214). It is chosen so as to make

$$\phi + \psi = 0$$

at the instant $t = 0$ when the motion first commences. If at this instant $\mathbf{OC}_0$ is identical with **OZ**, as will usually be the case, then the reference plane will be the downward vertical plane through $\mathbf{OC}_0$, since then $\phi = \psi = 0$. With this choice we have

$$\sigma = \int_0^t r\,dt = \phi + \psi. \tag{3·3·9}$$

Put
$$\sigma_C = \phi_C + \psi_C. \tag{3·3·10}$$

Then $d\sigma_C/dt$ represents the angular velocity of the axes of inertia **OA**, **OB** about **OC**. The total angular velocity of the axes of inertia is

$$\boldsymbol{\omega}_C = \frac{1}{\alpha_C}\frac{d\alpha_C}{dt}\mathbf{OC}_0 \times \mathbf{OC} + \frac{d\phi_C}{dt}\mathbf{OC}_0 + \frac{d\psi_C}{dt}\mathbf{OC}. \tag{3·3·11}$$

The *complex rate of turn* of the axis is obtained by differentiating (3·3·2), namely

$$\frac{d\zeta}{dt} = \left(\frac{d\theta}{dt} + i\frac{d\psi}{dt}\theta\right) e^{i\psi} = \omega\, e^{i\psi'} \tag{3·3·12}$$

say. We also call $d\zeta/dt$ the (complex) *cross-spin*.

When the equations of motion are solved, expressions are given for Z, Ξ and $d\zeta/dt$. From these ζ may be obtained from (3·3·3), and the *complex linear deviation* of the trajectory from the formula

$$D\, e^{i\Upsilon} = \int_{s_0}^{s} Z\, ds. \tag{3·3·13}$$

Here $D\cos\Upsilon$ is the drop of the centre of gravity below and perpendicular to the line of projection (at launch), and $D\sin\Upsilon$ is the lateral deviation to the left as viewed from the rear.

Other quantities used are the *complex cross-wind*

$$w_1 = w\, e^{i\xi_1}\sin\xi, \tag{3·3·14}$$

and
$$\eta = V\Xi + w_1. \tag{3·3·15}$$

In referring to the quantities defined by (3·3·1, 2, 3, 12, 13, 14) the word 'complex' will usually be omitted.

3·31. *Mapping on the Argand diagram*

If any of the complex quantities defined in § 3·3 above are plotted at a number of instants during burning, on the Argand diagram, a curve can be drawn which is useful in illustrating the variation of the quantity concerned. Since **OY** is to the left when viewed by an observer looking along the direction of projection **OZ**, and **OX** is downwards, it is usually convenient to reflect the Argand diagram of the complex plane in the y-axis, and then rotate in a positive (anticlockwise) direction through one right angle, in order to obtain a suitable picture.

Thus the curve C_1 whose cartesian co-ordinates are given by

$$x = -\Theta\sin\Psi, \quad y = -\Theta\cos\Psi$$

shows the angular deviation Θ and the orientation Ψ of the tangent to the trajectory about the line of projection as viewed from behind. The x- and y-axes of the graph correspond to the directions opposite to **OY** and **OX** respectively.

Similarly, the curve C_2 whose cartesian co-ordinates are given by

$$x = -\delta\sin\chi, \quad y = -\delta\cos\chi$$

shows the yaw δ and the orientation of the rocket's axis $\mathbf{OC}_0$ about the instantaneous direction of the tangent to the trajectory **OT** as viewed from behind.

Curves of the forms C_1 and C_2 are shown in figures 6 to 15. Similar graphs may be drawn to illustrate the motion of the axis (ζ) and its rate of turn ($d\zeta/dt$), or the linear deviation $D\,\mathrm{e}^{i\Upsilon}$.

3·32. *Complex vectors*

When the equations of motion have been set up in general vector form it is necessary to resolve them along certain suitable directions in order to reduce them to forms convenient for solution. The motion in the forward direction, and about the rocket's axis, can be separated out by resolving along **OZ** or $\mathbf{OC}_0$. The motion at right angles to **OZ** or $\mathbf{OC}_0$ is found most conveniently by combining one perpendicular component with i times the other perpendicular component, i.e. by forming the scalar product with $\mathbf{OX}+i\mathbf{OY}$ or $\mathbf{OA}_0+i\mathbf{OB}_0$.

Thus, for example,

$$\begin{aligned}(\mathbf{OX}+i\mathbf{OY})\,.\,\mathbf{V} &= V(\mathbf{OX}+i\mathbf{OY})\,.\,\mathbf{OT} = V(\Theta\cos\Psi+i\Theta\sin\Psi)\\ &= VZ.\end{aligned}$$

The following relations are of help in forming these scalar products. **E** denotes any vector.

$$(\mathbf{OX}+i\mathbf{OY})\,.\,\mathbf{E} = \mathrm{e}^{i\psi}\,(\mathbf{OA}_0+i\mathbf{OB}_0)\,.\,\mathbf{E}+\zeta\mathbf{E}\,.\,\mathbf{OZ}, \tag{3·32·1}$$

$$(\mathbf{OX}+i\mathbf{OY})\,.\,(\mathbf{OZ}\times\mathbf{E}) = i(\mathbf{OX}+i\mathbf{OY})\,.\,\mathbf{E}, \tag{3·32·2}$$

$$(\mathbf{OA}_0+i\mathbf{OB}_0)\,.\,(\mathbf{OC}_0\times\mathbf{E}) = i(\mathbf{OA}_0+i\mathbf{OB}_0)\,.\,\mathbf{E}. \tag{3·32·3}$$

A list of components of various vectors is now given. In it P denotes any one of the six letters G, K, L, M, N, R. In setting up the equations only the components along the fixed directions **OZ** and $\mathbf{OX}+i\mathbf{OY}$ are required.

	components along			
vector	**OX**	**OY**	**OZ**	$\mathbf{OX}+i\mathbf{OY}$
$\mathbf{OA}_0$	$\cos\psi$	$\sin\psi$	$-\theta$	$\mathrm{e}^{i\psi}$
$\mathbf{OB}_0$	$-\sin\psi$	$\cos\psi$	0	$i\,\mathrm{e}^{i\psi}$
$\mathbf{OC}_0$	$\theta\cos\psi$	$\theta\sin\psi$	1	ζ
OT	$\Theta\cos\Psi$	$\Theta\sin\Psi$	1	Z
OV	$-\cos\alpha$	0	$\sin\alpha$	$-\cos\alpha$
OW	$\sin\xi\cos\xi_1$	$\sin\xi\sin\xi_1$	$\cos\xi$	$\sin\xi\,\mathrm{e}^{i\xi_1}$

	components along				
vector	$\mathbf{OA}_0$	$\mathbf{OB}_0$	$\mathbf{OC}_0$	$\mathbf{OX}+i\mathbf{OY}$	**OZ**
OX	$\cos\psi$	$-\sin\psi$	$\theta\cos\psi$	1	0
OY	$\sin\psi$	$\cos\psi$	$\theta\sin\psi$	i	0
OZ	$-\theta$	0	1	0	1
$\mathbf{OX}+i\mathbf{OY}$	$\mathrm{e}^{i\psi}$	$i\,\mathrm{e}^{i\psi}$	ζ	0	0
$\boldsymbol{\Omega}$	$-\dot{\psi}\theta$	$\dot{\theta}$	r	$i(d\zeta/dt)+r\zeta$	r
$\boldsymbol{\omega}$	$-\dot{\psi}\theta$	$\dot{\theta}$	0	$i(d\zeta/dt)$	0
OP	$\alpha_P\cos(\phi+\phi_P)$	$\alpha_P\sin(\phi+\phi_P)$	1	$\alpha_P\,\mathrm{e}^{i(\sigma+\phi_P)}+\zeta$	1
OB	$-\sin(\phi+\sigma_C)$	$\cos(\phi+\sigma_C)$	$\alpha_C\sin\psi_C$	$i\,\mathrm{e}^{i(\sigma+\sigma_C)}$	$\alpha_C\sin\psi_C+\theta\sin(\phi+\sigma_C)$
OC	$\alpha_C\cos(\phi+\phi_C)$	$\alpha_C\sin(\phi+\phi_C)$	1	$\alpha_C\,\mathrm{e}^{i(\sigma+\phi_C)}+\zeta$	1
$\dot{\mathbf{OC}}$			0	$[(d\alpha_C/dt)+i\alpha_C(d\phi_C/dt)]\,\mathrm{e}^{i(\sigma+\phi_C)}$	0
$\boldsymbol{\omega}_C$			$d\sigma_C/dt$	$[\alpha_C(d\psi_C/dt)+i(d\alpha_C/dt)]\,\mathrm{e}^{i(\sigma+\phi_C)}+(d\sigma_C/dt)\,\zeta$	$d\sigma_C/dt$

The components of the following vectors occurring in equations (2·5·11, 12) are also required:

vector	components along OX + *i*OY	OZ
$\mathbf{OR}\times\mathbf{ON}'$	$i\alpha_N\,e^{i(\sigma+\phi_N)}-i\alpha_R\,e^{i(\sigma+\phi_R)}$	0
$\mathbf{OG}\times\mathbf{OM}'$	$i\alpha_M\,e^{i(\sigma+\phi_M)}-i\alpha_G\,e^{i(\sigma+\phi_G)}$	0
$\mathbf{\Omega}\times\mathbf{ON}'$	$d\zeta/dt+ir\alpha_N\,e^{i(\sigma+\phi_N)}$	0
$\mathbf{\Omega}\times\mathbf{OG}'$	$d\zeta/dt+ir\alpha_G\,e^{i(\sigma+\phi_G)}$	0
$\mathbf{ON}'\times(\mathbf{\Omega}\times\mathbf{ON}')$	$i(d\zeta/dt)-r\alpha_N\,e^{i(\sigma+\phi_N)}$	0
$\mathbf{\Omega}\times(\mathbf{OM}'\times\mathbf{OG}')$	$r\{\alpha_M\,e^{i(\sigma+\phi_M)}-\alpha_G\,e^{i(\sigma+\phi_G)}\}$	0

These are used in conjunction with

$$\mathbf{W}=-W\mathbf{OR}, \tag{3·32·4}$$

$$\mathbf{R}_N=-l\mathbf{ON}',\quad \mathbf{R}_M=-l\mathbf{OM}', \tag{3·32·5}$$

$$\mathbf{n}=\mathbf{OG}',\quad \mathbf{k}=\mathbf{OK}. \tag{3·32·6}$$

3·4. Symmetry of order *n*

It is convenient to introduce this concept here. A surface is said to possess symmetry of the *n*th order about an axis, which is called the axis of symmetry, when the following three conditions are satisfied:

(i) *n* is a positive integer.

(ii) If r, θ and z are the cylindrical co-ordinates of any point of the surface with respect to the axis of symmetry, i.e. r denotes distance from, θ orientation with respect to, and z distance along the axis, then the point whose co-ordinates are r, $\theta+2\pi/n$, z also belongs to the surface.

(iii) *n* is the greatest integer satisfying (ii).

In general terms this means that the surface can be divided into *n*, but not more than *n*, identical sectors about the axis of symmetry. Clearly every surface possesses symmetry of at least the first order about any axis, so that the concept has no value unless *n* is greater than unity.

It is clear that the concept may be extended to volumes and to any two- or three-dimensional bodies. When applied to a body it is assumed that the corresponding parts of the different sections are identical in density as well as in position. It is easy to establish the following result:

If a body or surface possesses symmetry of order greater than 2 about an axis, then this axis is a principal axis of inertia, and the moments of inertia of the body or surface about any two axes perpendicular to this axis and passing through it are equal.

3·5. The force system

The forces and couples which act on the rocket after launch fall into three groups, (i) forces and couples due to the combustion of the propellant, (ii) the force of gravity, and (iii) aerodynamic† forces and couples due to the action of the air on the surface of the projectile.

† There is also an aerostatic force which is quite trivial for the rocket and which we neglect altogether.

The first group has already been considered in § 2. Its most important members are the thrust $Q\mathbf{W}$, and the torque $\mathbf{G}_R$ when the projectile is spun by the gases.

The force of gravity requires little comment. It has the form $m\mathbf{g}$, where g is the acceleration due to gravity, and $\mathbf{g} = -g\mathbf{OV}$.

The remainder of the present section (§ 3·5) is devoted to the discussion of the third group of forces. In the notation of § 2·4, $\mathbf{L}$ and $\mathbf{M}$ denote respectively the force and couple (about O) due to the forces of groups (ii) and (iii). Thus we may write

$$\mathbf{L} = m\mathbf{g} + \mathbf{L}_A, \quad \mathbf{M} = \mathbf{M}_A,$$

where $\mathbf{L}_A$ and $\mathbf{M}_A$ are the force and couple due to the aerodynamic forces alone.

The specification of $\mathbf{L}_A$ and $\mathbf{M}_A$ is complicated (*a*) by the presence and effect upon the air flow of the burning gases ejected from the rocket, and (*b*) by possible asymmetries in design of the projectile such as curvature of the tube, distorted fins, etc.

Suppose that $\mathbf{L}_S$ and $\mathbf{M}_S$ represent the aerodynamic force and couple which would act on the projectile under the same conditions of motion (velocity, spin, yaw, rate of yawing, etc.) when no gases are being ejected from the exit plane, i.e. which act upon the *equivalent shell*. In order to tackle the problem of estimating $\mathbf{L}_A$ and $\mathbf{M}_A$ the following assumption is made:

A 2. *The gases ejected from the exit plane of the rocket during burning have no effect upon the air stream except at and behind the exit plane, and the force* $\mathbf{L}_A$ *and couple* $\mathbf{M}_A$ *are related to those* $\mathbf{L}_S$ *and* $\mathbf{M}_S$ *acting on the equivalent shell in the following manner:*

$$\mathbf{L}_A = \mathbf{L}_S + \mathbf{L}_e, \quad \mathbf{M}_A = \mathbf{M}_S + \mathbf{M}_e, \tag{3·5·1}$$

where

$$\mathbf{L}_e = \int_{S_0} \mathbf{p}\, dS, \quad \mathbf{M}_e = \int_{S_0} \mathbf{R} \times \mathbf{p}\, dS. \tag{3·5·2}$$

Here the notation employed is that of § 2, the vector $\mathbf{R}$ denoting distance from the centre of gravity, and the vector

$$\mathbf{p} = p\mathbf{k} = p\mathbf{OK}$$

denoting the difference at the exit plane between the gas pressure and the air pressure on the projectile.† In effect $\mathbf{L}_e$ and $\mathbf{M}_e$ are corrections for gas pressure and absence of base drag at the exit plane.

Assumption A 2 clearly cannot be expected to hold for rockets whose nozzles are not situated at the rear of the projectile. For normal rockets where the exit plane is the part of the projectile farthest to the rear the assumption is probably justified. The form of $\mathbf{L}_e$ and $\mathbf{M}_e$ and the magnitude of $\mathbf{p}$ are discussed in § 3·55.

At this point it is convenient to introduce a number of definitions.

Definition 1. *The axis of the equivalent shell* $\mathbf{OC}_0$ *is defined to be the direction close to that joining the centre of gravity O to the tip of the nose, such that when the projectile is moving with a constant velocity along a straight line in the direction* $\mathbf{OC}_0$, *then*

$$\mathbf{L}_S \times \mathbf{OC}_0 = 0, \quad \mathbf{M}_S \times \mathbf{OC}_0 = 0.$$

† If we had not made the assumption that S_0 was plane then $\mathbf{p}$ might have been a function of $\mathbf{R}$. Thus in the case where a multiple nozzle system is used and the exit plane of each individual nozzle is perpendicular to its axis the couple may not be negligible, as it is shown to be in § 3·55. The modifications necessary to include this case are not difficult to make and amount, in effect, to a change in the definition of $\mathbf{G}_R$.

The following assumption is made in this connexion:

A 3. $\mathbf{OC}_0$ *exists, is unique, and is fixed in direction relative to the shell.*

It is possible that two different directions $\mathbf{OC}_0^{(1)}$ and $\mathbf{OC}_0^{(2)}$ may exist for which

$$\mathbf{L}_S \times \mathbf{OC}_0^{(1)} = 0 \quad \text{and} \quad \mathbf{M}_S \times \mathbf{OC}_0^{(2)} = 0.$$

We shall not consider this case. It would, however, be possible to take it into account by modifying the theory, although the treatment would then become considerably more complicated; the direction of $\mathbf{OC}_0$ would in this case be chosen to be such that $\mathbf{M}_S \times \mathbf{OC}_0 = 0$, since the aerodynamic couple is of greater importance than the cross force.

DEFINITION 2. *The rocket's axis* $\mathbf{OC}_0$ *is defined to be that of the equivalent shell.*

The choice of axis which we have made is rather artificial in some ways, since it is not amenable to very precise measurement in practice. On the other hand, all definitions in terms of the geometrical form of the rocket† are distinctly unsatisfactory owing to the considerable asymmetries which occur in practice.

DEFINITION 3. *The equivalent shell is said to be an ideal shell if the following two conditions are satisfied:*

(*a*) *Its external surface possesses symmetry of order greater than* 2 *about an axis of symmetry which passes through the centre of gravity* (see § 3·4).

(*b*) *Reflexion of the external surface about any plane through the axis of symmetry is equivalent to a rotation about the axis of symmetry.*

The condition (*b*) excludes shell with offset fins, for example.

It follows from condition (*a*) that the axis of symmetry of an ideal shell is identical with its axis $\mathbf{OC}_0$ (as defined by Definition 1), and from (*b*) that $\mathbf{M}_S . \mathbf{OC}_0 = 0$ when the spin about the axis is zero. It is, of course, not necessary for $\mathbf{OC}$ to coincide with $\mathbf{OC}_0$ or for A and B to be equal.

By virtue of Assumption A 2 it remains to specify the force $\mathbf{L}_S$ and couple $\mathbf{M}_S$ acting upon the equivalent shell. In order to do this, we first investigate the force system for an ideal shell moving in still air, and then examine what additional modifications are necessary (i) when there is a wind, (ii) when the projectile (i.e. the equivalent shell) possesses offset fins, and (iii) when the projectile is slightly asymmetrical. The results if these investigations are collated in § 3·56, where the full aerodynamic force and couple system acting on the rocket is stated.

3·51. *The aerodynamic force system on an ideal shell moving in still air*

Let $\mathbf{L}_I$ and $\mathbf{M}_I$ be respectively the total force and couple about the centre of gravity acting on the ideal shell. Let $\mathbf{R}$ and $\mathbf{\Gamma}$ be the components of $\mathbf{L}_I$ and $\mathbf{M}_I$ along the axis $\mathbf{OC}_0$. Write

$$\mathbf{L}_I = \mathbf{R} + \mathbf{F}, \quad \mathbf{M}_I = \mathbf{\Gamma} + \mathbf{G}, \tag{3·51·1}$$

where $\mathbf{F}$ and $\mathbf{G}$ represent the combined lateral force and couple acting perpendicular to the axis. By Definition 1, both $\mathbf{F}$ and $\mathbf{G}$ vanish when the projectile is moving along $\mathbf{OC}_0$ in steady motion with no yaw or rate of turn.

† For example, (i) the line joining the nose tip to the centre of the exit plane or (ii) the axis of the nozzle. These two lines need not even pass through the centre of gravity.

It is convenient to split the force **F** and couple **G** each into four parts:

$$\mathbf{F} = \mathbf{L}_1+\mathbf{L}_2+\mathbf{L}_3+\mathbf{L}_4, \tag{3·51·2}$$

$$\mathbf{G} = \mathbf{M}_1+\mathbf{M}_2+\mathbf{M}_3+\mathbf{M}_4. \tag{3·51·3}$$

Thus the total aerodynamic force and couple system consists of five forces and five couples. These ten components are described and named in §§ 3·512 to 3·516, i.e. they are attributed to various 'causes' such as cross-velocity, cross-spin, etc. The method by which the eight components of **F** and **G** are deduced from the Assumption A 1 and from Definition 3 is briefly sketched in the following subsection § 3·511, which may be omitted by the reader if desired.

3·511. *Deduction of the lateral force system*

The material of this subsection is due, in the main, to Nielsen & Synge (1946, § 4) and to Kelley, McShane & Reno (1949?). It is included here for the sake of completeness, but in an abbreviated form. The notation used is employed only in the present subsection and in §§ 3·517 and 3·54.

Let F_1, F_2 be the components of **F** along any two perpendicular directions fixed in the body and lying in a plane normal to $\mathbf{OC}_0$. Let G_1, G_2, V_1, V_2, ω_1 and ω_2 be the corresponding components of **G**, **V** and $\boldsymbol{\omega}$. The following assumptions are made.

A 4. *When* $V_1 = V_2 = 0$, $\omega_1 = \omega_2 = 0$, *the lateral force and couple components* F_1, F_2, G_1 *and* G_2 *vanish or can be neglected.*

A 5. *For* $i = 1, 2, j = 1, 2,$ $\quad \dfrac{\partial F_i}{\partial V_j}, \; \dfrac{\partial F_i}{\partial \omega_j}, \; \dfrac{\partial G_i}{\partial V_j}, \; \dfrac{\partial G_i}{\partial \omega_j}$

exist and are continuous for the ranges of linear and angular velocity considered.

In simple language these assumptions assert that, to the order of approximation allowable by Assumption A 1, each of the four quantities F_1, F_2, G_1 and G_2 is expressible as a linear form in the four quantities V_1, V_2, ω_1 and ω_2. These linear forms do not possess constant terms, by Assumption A 4, so that the lateral force and couple system is describable by sixteen real coefficients. It is not assumed at this stage that these sixteen coefficients are independent of V_1, V_2, ω_1 and ω_2 and their derivatives.

So far none of the special properties of the ideal shell have been used. We now make use of the first property, namely, condition (*a*) of Definition 3. Let n be the order of symmetry of the shell. Then, since the external form and centre of gravity are unaltered when the body is rotated through $2\pi/n$ about its axis of symmetry, the force system must be unaffected by such a rotation. When this is expressed in mathematical form the number of different coefficients reduces from sixteen to eight provided that (see chapter 12, Kelley, McShane & Reno 1949?)

$$e^{4\pi i/n} \neq 1,$$

i.e. provided that $n > 2$, and the lateral components can then be expressed in the simple forms

$$F^* = F_1+iF_2 = f_1 V^* + f_2\omega^*, \tag{3·511·1}$$

$$G^* = G_1+iG_2 = g_1 V^* + g_2\omega^*, \tag{3·511·2}$$

where f_1, f_2, g_1 and g_2 are *complex* quantities, and

$$V^* = V_1+iV_2, \quad \omega^* = \omega_1+i\omega_2. \tag{3·511·3}$$

The value of G^* depends, of course, upon the position of the point about which moments are taken, in this case the centre of gravity. If a different point O' at a distance z on the axis forward of O is chosen, it is easy to show that the new coefficients (denoted by dashes) of the new 'complex' lateral force $F^{*\prime}$ and couple $G^{*\prime}$ are given by

$$f_1' = f_1, \quad f_2' = f_2 + izf_1, \quad g_1' = g_1 - izf_1, \quad g_2' = g_2 + iz(g_1 - f_2) + z^2 f_1. \qquad (3{\cdot}511{\cdot}4)$$

These equations may be regarded as a transformation $T(z)$. It is easy to see that $f_1, f_2 + g_1$ and $f_1 g_2 - f_2 g_1$ are invariants of the transformation, and that any such invariant is a function of these three expressions. Accordingly, if the four complex coefficients f_1, f_2, g_1 and g_2 are connected by any functional relation, it can only be of the form†

$$\Phi(f_1, f_2 + g_1, f_1 g_2 - f_2 g_1) = 0.$$

Remarks on the Assumptions A 4 *and* A 5. Assumptions similar to A 4 and A 5 are stated or are implied in the work of Nielsen & Synge (1946) and of Kelley, McShane & Reno (1949?). The first is equivalent to assuming that the lateral forces and couples which act in unsteady motion are of the same character as those which act when the motion is steady, i.e. it is assumed that although the coefficients f_1, f_2, g_1 and g_2 may possibly be affected by the rate of change‡ of V, r, V_1, V_2, ω_1 and ω_2, no extra forces and couples are required to describe the motion. It would, of course, be possible to cater for such extra forces in the theory. For example, the effect of rate of change of cross-velocity and cross-spin could be allowed for by the addition of terms $f_3(dV^*/dt) + f_4(d\omega^*/dt)$ and $g_3(dV^*/dt) + g_4(d\omega^*/dt)$ to the right-hand sides of equations (3·511·1) and (3·511·2) respectively. This would increase the number of lateral forces and couples from eight to sixteen, and would contribute terms in dZ/dt and $d^2\zeta/dt^2$ to the general equations. In view of the lack of information on unsteady motion, it is considered that such an extension of the force and couple system is not justifiable at present.

3·512. *Axial drag and torque*

The axial drag is due to the resistance of the air to the projectile's forward motion. It is denoted by

$$\mathbf{R} = -R\mathbf{OC}_0 = -R_A V^2 \mathbf{OC}_0. \qquad (3{\cdot}512{\cdot}1)$$

The dependence of the coefficient R_A upon velocity, etc., and the relation between $\mathbf{R}$ and the ordinary tangential drag used in standard trajectory calculations is discussed in § 3·518.

The axial torque exists only when the projectile is spinning about its axis, and acts in a direction opposing the spin. It is denoted by

$$\mathbf{\Gamma} = -\Gamma\mathbf{OC}_0 = -\Gamma_A aVr\mathbf{OC}_0. \qquad (3{\cdot}512{\cdot}2)$$

For finless projectiles $\mathbf{\Gamma}$ is due almost entirely to skin friction. When the projectile has fins, there will be a considerable contribution to $\mathbf{\Gamma}$ from the normal force on each fin due to the relative motion of the fin blades to the air stream caused by the axial spin (see § 3·53).

† The assumption $f_2 + g_1 = 0$ (i.e. in the notation of § 3·517, $k' = kd_1$, $k'_M = k_M d_3$) is one which would be consistent and which might, at first sight, be expected to hold, especially for finned rockets. It is not, however, borne out by experimental evidence, which indicates that the cross-spin forces are larger than this relation would imply. This evidence is also corroborated by theoretical investigations of the analogous two-dimensional case which have shown that a considerable part of the cross-spin force and couple is due to the vorticity which is developed in the wake as a result of the unsteady motion of the aerofoil.

‡ For the effect of linear and angular acceleration on the coefficients see § 3·518.

3·513. *Force and couple due to cross-velocity*

By cross-velocity is meant velocity perpendicular to the axis. We denote the force by L_1 and the couple (about the centre of gravity) by M_1. Their directions are given by

$$\mathbf{L}_1 = L_1 \mathbf{OS}' = L_1 \operatorname{cosec} \delta (\mathbf{OC}_0 \cos \delta - \mathbf{OT}), \qquad (3·513·1)$$

and

$$\mathbf{M}_1 = M_1 \mathbf{OS} = M_1 \operatorname{cosec} \delta \, \mathbf{OC}_0 \times \mathbf{OT}, \qquad (3·513·2)$$

so that $\mathbf{L}_1$ acts in the plane of yaw, and $\mathbf{M}_1$ about an axis normal to this plane, M_1 being positive when the moment is stabilizing. We write

$$L_1 = kV^2\delta, \quad M_1 = kd_1 V^2\delta. \qquad (3·513·3)$$

The quantity d_1 may be regarded as defining the distance of the associated centre of pressure to the rear of the centre of gravity.

In the literature the force $\mathbf{L}_1$ is also called the normal force, and $\mathbf{M}_1$ the righting moment or stabilizing lift moment. The relation between $\mathbf{L}_1$ and the lift is given in § 3·518.

3·514. *Force and couple due to cross-spin*

By cross-spin is meant the rate of turn $\boldsymbol{\omega}$ of the axis. The force and couple are denoted by $\mathbf{L}_2$ and $\mathbf{M}_2$ respectively, where

$$\mathbf{L}_2 = L_2 \omega^{-1} \boldsymbol{\omega} \times \mathbf{OC}_0 \qquad (3·514·1)$$

and

$$\mathbf{M}_2 = -M_2 \omega^{-1} \boldsymbol{\omega}, \qquad (3·514·2)$$

so that $\mathbf{L}_2$ acts in the plane of yawing, i.e. in the instantaneous plane in which the axis is moving about the centre of gravity; $\mathbf{M}_2$ acts about an axis normal to this plane and $\mathbf{M}_2$ is positive when the couple resists yawing. Write

$$L_2 = k' V\omega, \quad M_2 = k'd_2 V\omega. \qquad (3·514·3)$$

The form and magnitude of the coefficients k' and $k'd_2$ are discussed in § 3·518.

3·515. *Magnus force and couple due to cross-velocity*

This force and couple are due to the lateral movement of the projectile in the air stream and the axial spin. The spin causes a circulation to take place round the projectile and thus produces a force $\mathbf{L}_3$ at right angles to the plane of yaw and a corresponding couple $\mathbf{M}_3$. Their directions are given by

$$\mathbf{L}_3 = L_3 \operatorname{cosec} \delta \, \mathbf{OC}_0 \times \mathbf{OT}, \qquad (3·515·1)$$

and

$$\mathbf{M}_3 = M_3 \mathbf{OS} \times \mathbf{OC}_0 = M_3 \operatorname{cosec} \delta (\mathbf{OT} - \mathbf{OC}_0 \cos \delta). \qquad (3·515·2)$$

We write

$$L_3 = k_M rV\delta, \quad M_3 = k_M d_3 rV\delta. \qquad (3·515·3)$$

The quantity d_3 determines the distance of the associated centre of pressure behind the centre of gravity.

For a discussion of the form and magnitude of the coefficients k_M and $k_M d_3$ see § 3·518.

3·516. *Magnus force and couple due to cross-spin*†

A similar Magnus force $\mathbf{L}_4$ and couple $\mathbf{M}_4$ are caused by the angular velocity of the axis. Their directions are given by

$$\mathbf{L}_4 = -L_4 \omega^{-1} \boldsymbol{\omega}, \qquad (3·516·1)$$

† The force $\mathbf{L}_4$ and couple $\mathbf{M}_4$ are not included in the force system of Fowler *et al.* (1920). Their introduction is due to Nielsen & Synge (1946) who found that they were necessary in order that the force system might be consistent and independent of the position of the point about which moments are taken. They are also responsible for the introduction of the cross-spin force $\mathbf{L}_2$ into shell ballistics; this force had, however, been taken into consideration in rocket ballistics previous to the work of Nielsen & Synge.

and $$\mathbf{M}_4 = -M_4\omega^{-1}\boldsymbol{\omega}\times\mathbf{OC}_0. \tag{3·516·2}$$

We write $$L_4 = k'_M r\omega, \quad M_4 = k'_M d_4 r\omega. \tag{3·516·3}$$

The coefficients k'_M and $k'_M d_4$ are discussed in § 3·518.

3·517. *Summary of the force system on an ideal shell*

From the formulae given in the preceding subsections we have, by Assumption A 1,

$$\mathbf{L}_I = -R\mathbf{OC}_0 + \frac{L_1}{\delta}(\mathbf{OC}_0 - \mathbf{OT}) + \frac{L_2}{\omega}\boldsymbol{\omega}\times\mathbf{OC}_0 + \frac{L_3}{\delta}\mathbf{OC}_0\times\mathbf{OT} - \frac{L_4}{\omega}\boldsymbol{\omega}, \tag{3·517·1}$$

and $$\mathbf{M}_I = -\Gamma\mathbf{OC}_0 + \frac{M_1}{\delta}\mathbf{OC}_0\times\mathbf{OT} - \frac{M_2}{\omega}\boldsymbol{\omega} - \frac{M_3}{\delta}(\mathbf{OC}_0 - \mathbf{OT}) - \frac{M_4}{\omega}\boldsymbol{\omega}\times\mathbf{OC}_0. \tag{3·517·2}$$

On substituting for $L_1, L_2, \ldots, M_3, M_4$ in terms of the eight coefficients $k, k', k_M, k'_M, d_1, d_2, d_3$ and d_4, and resolving along the directions $\mathbf{OC}_0$, $\mathbf{OA}_0 + i\mathbf{OB}_0$, $\mathbf{OZ}$ and $\mathbf{OX}+i\mathbf{OY}$, we obtain, by Assumption A 1,

$$\mathbf{L}_I.\mathbf{OC}_0 = \mathbf{L}_I.\mathbf{OZ} = -R = -R_A V^2, \tag{3·517·3}$$

$$\mathbf{M}_I.\mathbf{OC}_0 = \mathbf{M}_I.\mathbf{OZ} = -\Gamma = -\Gamma_A aVr, \tag{3·517·4}$$

$$\mathbf{L}_I.(\mathbf{OA}_0 + i\mathbf{OB}_0) = \mathrm{e}^{-i\psi}\{(kV - ik_M r)\,V\Xi - (ik'V + k'_M r)\,i(d\zeta/dt)\}, \tag{3·517·5}$$

$$\mathbf{M}_I.(\mathbf{OA}_0 + i\mathbf{OB}_0) = \mathrm{e}^{-i\psi}\{-(ikd_1 V + k_M d_3 r)\,V\Xi + (-k'd_2 V + ik'_M d_4 r)\,i(d\zeta/dt)\}, \tag{3·517·6}$$

$$\begin{aligned}\mathbf{L}_I.(\mathbf{OX}+i\mathbf{OY}) &= \frac{L_1 - iL_3}{\delta}\Xi + \frac{L_2 - iL_4}{\omega}\frac{d\zeta}{dt} - R\zeta \\ &= (kV - ik_M r)\,V\Xi + (k'V - ik'_M r)\,(d\zeta/dt) - R\zeta,\end{aligned} \tag{3·517·7}$$

and $$\begin{aligned}\mathbf{M}_I.(\mathbf{OX}+i\mathbf{OY}) &= -\frac{iM_1 + M_3}{\delta}\Xi - \frac{iM_2 + M_4}{\omega}\frac{d\zeta}{dt} - \Gamma\zeta \\ &= -(ikd_1 V + k_M d_3 r)\,V\Xi - (ik'd_2 V + k'_M d_4 r)\,(d\zeta/dt) - \Gamma\zeta.\end{aligned} \tag{3·517·8}$$

The connexion between the coefficients of $-V\Xi$ and $i(d\zeta/dt)$ in (3·517·5, 6) and those of V^* and ω^* in (3·511·1, 2) is now clear. Take $\mathbf{OA}_0$ and $\mathbf{OB}_0$ to be the instantaneous positions of the perpendiculars to $\mathbf{OC}_0$ about which $\mathbf{F}$, $\mathbf{G}$, $\mathbf{V}$ and $\boldsymbol{\omega}$ are resolved in § 3·511. Then we have

$$\left.\begin{aligned} F^* &= \mathbf{F}.(\mathbf{OA}_0 + i\mathbf{OB}_0) = \mathbf{L}_I.(\mathbf{OA}_0 + i\mathbf{OB}_0), \\ G^* &= \mathbf{G}.(\mathbf{OA}_0 + i\mathbf{OB}_0) = \mathbf{M}_I.(\mathbf{OA}_0 + i\mathbf{OB}_0), \\ V_1 + iV_2 &= \mathbf{V}.(\mathbf{OA}_0 + i\mathbf{OB}_0) = -V\Xi\,\mathrm{e}^{-i\psi}, \end{aligned}\right\} \tag{3·517·9}$$

and $$\omega_1 + i\omega_2 = \boldsymbol{\omega}.(\mathbf{OA}_0 + i\mathbf{OB}_0) = i(d\zeta/dt)\,\mathrm{e}^{-i\psi}, \tag{3·517·10}$$

so that $$\left.\begin{aligned} f_1 &= -kV + ik_M r, & f_2 &= -ik'V - k'_M r, \\ g_1 &= ikd_1 V + k_M d_3 r, & g_2 &= -k'd_2 V + ik'_M d_4 r. \end{aligned}\right\} \tag{3·517·11}$$

With the help of these four relations we can write the formulae (3·511·4) in terms of the eight real aerodynamic coefficients and so display their interdependence. Let asterisks denote values of $k, kd_1, k', k'd_2, k_M, k_M d_3, k'_M$ and $k'_M d_4$ when the moments of the aerodynamic forces are taken not about the centre of gravity but about a point on the axis a distance z forward of it. We use asterisks in place of the dashes employed in § 3·511 because of the dashes already occurring on certain of the symbols. Then, by equating real and imaginary parts in (3·511·4) and using (3·517·11), we obtain

$$\left.\begin{aligned} k^* &= k, & k'^* &= k'+zk, \\ k^*d_1^* &= kd_1+zk, & k'^*d_2^* &= k'd_2+z(kd_1+k')+z^2k, \end{aligned}\right\} \qquad (3{\cdot}517{\cdot}12)$$

and

$$\left.\begin{aligned} k_M^* &= k_M, & k_M'^* &= k_M'+zk_M, \\ k_M^*d_3^* &= k_Md_3+zk_M, & k_M'^*d_4^* &= k_M'd_4+z(k_Md_3+k_M')+z^2k_M. \end{aligned}\right\} \qquad (3{\cdot}517{\cdot}13)$$

3·518. *Remarks on the force system and the associated coefficients*

The five forces and five couples have been expressed in §§ 3·512 to 3·516 in terms of the velocity of the centre of gravity V, the yaw δ, the axial spin r, the cross-spin ω, and of the ten coefficients†

$$R_A,\ \Gamma_A;\quad k,\ k',\ k_M,\ k_M';\quad kd_1,\ k'd_2,\ k_Md_3,\ k_M'd_4. \qquad (3{\cdot}518{\cdot}1)$$

It is not intended to convey by these expressions that these ten coefficients are independent of V, δ, r and ω, but only to indicate the chief quantities upon which the forces and couples depend, and to display the approximate nature of this dependence. In general, the ten coefficients will depend‡ upon (i) the size and shape of the projectile, (ii) the density, temperature and viscosity of the air, (iii) V, δ, r and ω, (iv) the rate of change of V, δ, r and ω and their derivatives, and (v) the position of the centre of gravity of the projectile (which moves during burning). Of these the first group does not vary during the motion, and the second only very slowly. It is the third group which is of greatest importance in the majority of cases, and since δ, ω and $a_M r/V$ are small, by Assumption A 1, most of the variation of the coefficients is due to changes in the forward velocity V. When the velocity is subsonic the ten coefficients will, apart from the effects of groups (iv) and (v), usually be approximately constant, but may increase (or decrease) sharply as the velocity approaches and passes through that of sound; at supersonic velocities there is usually a tendency for the coefficients to decrease (or increase) slowly with increasing velocity.

The effect of group (iv), i.e. of unsteady motion, upon the coefficients is not known. It is thought that of the various factors affecting the coefficients, the linear acceleration of the rocket along its trajectory during burning is likely to be by far the most important.§

† The four moment arms d_1, d_2, d_3 and d_4 occur only in the combinations kd_1, $k'd_2$, k_Md_3 and $k_M'd_4$, and these last four quantities are therefore regarded as the relevant coefficients. It may happen, for example, that $k'=0$, but this does not necessarily imply that $k'd_2$ vanishes. Although the notation is open to criticism on this and other accounts, no confusion is likely to arise, and the present notation, which is based partly on tradition and partly on practical convenience, has been retained.

‡ This dependence is usually stated most conveniently in terms of non-dimensional quantities such as the Reynolds number and the Mach number.

§ A considerable amount of theoretical work has been done on the effect of unsteady motion on the forces acting on a two-dimensional aerofoil. See, for example, the work of Glauert and others as described in Glauert (1929) or Durand (1935, vol. 2, Chapter 5). For an aerofoil of this type the contributions to the lift and moment due to variable velocity would appear to be by no means negligible. Further theoretical work on the unsteady motion of such an aerofoil has been done more recently for both subsonic and supersonic velocities. However, for aerofoils of small aspect ratio, such as the rocket, no reliable information on the effect of acceleration is available. Nevertheless, it is natural to suppose that the effects will be similar in character to those worked out for the two-dimensional case. Thus it is to be expected that acceleration along the trajectory will increase both the lift and the stabilizing moment, and that this will be most marked near launch when the velocity is low.

No general statement can be made as to the effect of group (v) upon the coefficients, as this will depend very much upon the design of the individual rocket considered. For projectiles possessing a high propellant-total weight ratio considerable changes in the coefficients† may occur during burning unless the centre of gravity of the charge is close to the centre of gravity of the complete projectile.

At this stage it is not assumed that any of the ten coefficients or any combination of them is constant. In order to obtain approximate solutions convenient for numerical application, it will be necessary, however, to make assumptions of this kind later on. These assumptions are listed in § 3·6 together with comments upon the extent to which they are justified (see also § 6·1).

The drag. In work on aerodynamics and ballistics the drag is usually taken parallel to the air stream, i.e. in the direction opposite to **OT**, the tangent of the trajectory, and not parallel to the axis as has been assumed here. Because of Assumption A 1, however, the 'tangential drag' is

$$\mathbf{L}_I . \mathbf{OT} = -R,$$

so that, for small yaws and cross-spins, the axial and tangential drags are the same.

The lift. The lift is usually defined as being perpendicular to the air stream. The force system developed in the preceding pages does not include a lift, its place being taken by the normal force $\mathbf{L}_1$. For an unrotated rocket with no axial or cross-spin the relation between the lift $\mathbf{L}_1''$, the normal force, and the drag is

$$L_1'' = L_1 \cos\delta - R\sin\delta \doteqdot L_1 - R\delta. \qquad (3·518·2)$$

Accordingly, if

$$L_1'' = k'' V^2 \delta, \qquad (3·518·3)$$

we have

$$k'' = k - R_A. \qquad (3·518·4)$$

The order of magnitude of the various forces and couples. The eight lateral forces and couples L_i, M_i $(i = 1-4)$, are first-order quantities in δ and ω by definition. Similarly, because of Assumption A 1, Γ may be regarded as a first-order quantity in ar/V, since we may write it in the form

$$\Gamma = \Gamma_A V^2 \frac{ar}{V},$$

which is analogous to

$$M_1 = kd_1 V^2 \delta.$$

For the motion of a shell along the whole trajectory the resistance **R** cannot be regarded as a first-order quantity. However, during the burning period of the rocket it is small in comparison with the axial component of the thrust (see Assumption A 9 in § 3·55 below). Since we are only interested in the application of the theory to the period of burning of a rocket, we may regard **R** as being a first-order force.

With these conventions, it follows that first-order variations in V, r and the ten coefficients (3·518·1) will produce second-order variations in L_i, M_i, Γ and R which can be neglected under Assumption A 1.

In the equations of motion the lateral aerodynamic force and couple components of $\mathbf{L}_i$ and $\mathbf{M}_i$ occur side by side with 'jet' force and couple components of the orders of magnitude of

$$QW\zeta, \quad QWZ, \quad QW\alpha_P, \quad Qrl\alpha_P \quad \text{(forces)},$$
$$QWl\zeta, \quad QWlZ, \quad QWl\alpha_P, \quad Qrl^2\alpha_P \quad \text{(couples)},$$

† Not only the moment coefficients, but the force coefficients k' and k'_M will alter (see equations (3·511·4)).

where α_P is one of the angles defined in § 3·214. These forces and couples are also regarded as first-order forces and couples.

We have, accordingly, set up a scale by which it is possible to state which forces and couples are to be regarded as of the first order and which of the second order. Those of the second order are neglected under Assumptions A 1 and A 9. For example, $R\zeta$ is a second-order force according to our convention, since it is small in comparison with the first-order force R. It is scarcely necessary to remark that this division of forces and couples into groups of different orders of smallness cannot be infallible, and that cases will arise in practice where the conventions adopted do not apply, and where more careful examination of the relative magnitudes of the different forces and couples is therefore essential. It is, however, clear that, if a general theory is to be developed, some scale of magnitudes must be set up in order that the number of terms included in the equations may be kept within reasonable bounds.

3·52. *The effect of wind upon an ideal shell*

The presence of a wind alters the direction and magnitude of the resultant air stream. Relative to the air the projectile moves with a velocity

$$\mathbf{V}-w\mathbf{OW}. \tag{3·52·1}$$

The component of this velocity along the axis is

$$V_W = \mathbf{OC}_0.(\mathbf{V}-w\mathbf{OW}) = V-w\cos\xi_2 = V-w\cos\xi, \tag{3·52·2}$$

by Assumption A 1, since w/V and $\xi-\xi_2$ are first-order quantities. This is also, to our order of approximation, the total velocity. The component of the velocity perpendicular to the axis can be written in the complex form

$$\begin{aligned}(\mathbf{V}-w\mathbf{OW}).(\mathbf{OA}_0+i\mathbf{OB}_0) &= \mathrm{e}^{-i\psi}\{-V\Xi-w\sin\xi\,\mathrm{e}^{i\xi_1}+w\zeta\cos\xi\}\\ &= -V\mathrm{e}^{-i\psi}\left(\Xi+\frac{w}{V}\sin\xi\,\mathrm{e}^{i\xi_1}\right),\end{aligned} \tag{3·52·3}$$

by (3·32·1) and Assumption A 1, and V may be replaced by V_W in this formula since second-order quantities are neglected.

This shows that the wind has two effects, (i) that due to the additional forward velocity $-w\cos\xi$ of the projectile relative to the air, and (ii) that due to the additional cross-wind $w\sin\xi$. We examine first the effect of (i).

By Assumption A 1, w/V is small, and therefore the additional forward velocity of $-w\cos\xi$ relative to the air is equivalent to a first-order variation in V. By virtue of the remarks at the end of § 3·518 the effect of this extra velocity upon the lateral and axial forces and couples can be neglected during burning in the rocket application.

We now examine the effect of the cross-wind $w\sin\xi$. With the same assumptions as in § 3·511, this will introduce an ordinary force $\mathbf{L}_1'$ and couple $\mathbf{M}_1'$, and a Magnus force $\mathbf{L}_3'$ and couple $\mathbf{M}_3'$, due to the complex yaw

$$\frac{w}{V}\sin\xi\,\mathrm{e}^{i\xi_1} = \frac{w_1}{V}; \tag{3·52·4}$$

$\mathbf{L}_1'$ and $\mathbf{M}_3'$ act in the plane of this yaw, and $\mathbf{L}_3'$ and $\mathbf{M}_1'$ perpendicular to this plane. Write

$$\mathbf{L}_I' = \mathbf{L}_1'+\mathbf{L}_3', \tag{3·52·5}$$

$$\mathbf{M}_I' = \mathbf{M}_1'+\mathbf{M}_3', \tag{3·52·6}$$

so that $\mathbf{L}_I'$ and $\mathbf{M}_I'$ are the total force and couple due to the wind.

Then, as in § 3·517, we have

$$\mathbf{OC}_0 . \mathbf{L}'_I = \mathbf{OZ} . \mathbf{L}'_I = 0, \tag{3·52·7}$$

$$\mathbf{OC}_0 . \mathbf{M}'_I = \mathbf{OZ} . \mathbf{M}'_I = 0, \tag{3·52·8}$$

$$(\mathbf{OA}_0 + i\mathbf{OB}_0) . \mathbf{L}'_I = -w_1 \, e^{-i\psi} (-kV + ik_M r), \tag{3·52·9}$$

$$(\mathbf{OA}_0 + i\mathbf{OB}_0) . \mathbf{M}'_I = -w_1 \, e^{-i\psi} (ikd_1 V + k_M d_3 r), \tag{3·52·10}$$

$$(\mathbf{OX} + i\mathbf{OY}) . \mathbf{L}'_I = -w_1 (-kV + ik_M r), \tag{3·52·11}$$

and

$$(\mathbf{OX} + i\mathbf{OY}) . \mathbf{M}'_I = -w_1 (ikd_1 V + k_M d_3 r). \tag{3·52·12}$$

The coefficients k, k_M, kd_1 and $k_M d_3$ can be taken to be the same as for motion in still air, since the change in them due to the wind produces a second-order effect on the lateral components of the forces and couples, and this can be neglected.

3·53. *The effect of offset fins*

We consider a projectile with symmetry of order greater than 2 which is fitted with offset fins† in order to cause rotation or to prevent rotation being damped out. Fin assemblies may be of various types. We restrict ourselves to the consideration of the following common type.

The fin unit consists of a number of thin blades, called the fins, which are rigidly attached to the body of the projectile. Each fin surface forms part of a helical surface whose axis is the axis of symmetry of the projectile $\mathbf{OC}_0$. Such a surface has the representation

$$x = \rho \cos \theta, \quad y = \rho \sin \theta, \quad z = P\theta,$$

in terms of the variable parameters ρ and θ. Here x, y and z are measured along axes fixed in the body, the z-axis coinciding with $\mathbf{OC}_0$. The pitch‡ P is assumed to be constant. The inclination of each fin to the axis is

$$\Delta_\rho = \tan^{-1} \frac{\rho}{P}, \tag{3·53·1}$$

at a distance ρ from the axis. We make the following assumption.

A 6. *At each point of any fin Δ_ρ is small (say $\leqslant 15°$).*

It follows that we can put

$$\Delta_\rho = \frac{\rho}{P}. \tag{3·53·2}$$

For similar fins of the same area and span which are not offset ($\Delta_\rho = 0$), the projectile is an ideal shell, and the force system derived in §§ 3·51 and 3·52 applies. We regard the projectile with offset fins as a modification of this ideal shell and consider the effect upon the force and couple system. The projectile has the same order of symmetry as the ideal shell, and its force system can therefore be described by means of five forces and five couples. This is true, since the only place where condition (*b*) of Definition 3 was applied in §§ 3·51 and 3·52 was in the specification of the form of $\mathbf{\Gamma}$ in § 3·512.

† It follows that there are at least three fins.

‡ There are two definitions of pitch in common use. We use here the definition employed in mathematical literature, namely, the distance travelled per radian; to obtain the engineers' pitch this should be multiplied by 2π.

It follows that, since Δ_ρ is small, that the effect of offsetting the fins is of the second order on the lateral forces and couples and can therefore be neglected; the effect upon the axial drag is also of the second order in the application to the rocket, and is neglected (see end of § 3·518). The axial torque will, however, be different from that which acts in the case of the ideal shell. There will still be a damping couple which we can write

$$\mathbf{\Gamma} = -\Gamma_A aVr\mathbf{OC}_0, \tag{3·53·3}$$

as in the case of the ideal shell, since variations in Γ_A due to offsetting will produce second-order variations in $\mathbf{\Gamma}$.

The additional torque $\mathbf{\Gamma}_R$ due to offsetting may be regarded as being caused by the combined effect of the normal forces on each fin. Let $\rho = a_F$ be the distance from the axis of the point of application of each such fin force, and write $\Delta_\rho = \Delta_F$ at this point. Then we put

$$\mathbf{\Gamma}_R = \Gamma_F V^2 \Delta_F \mathbf{OC}_0, \tag{3·53·4}$$

where Γ_F is an aerodynamic coefficient which may be considered to be largely independent of Δ_F. The angle Δ_F is positive when the fins are inclined so as to impose clockwise rotation when viewed from the rear. When the fins are not offset $\Delta_F = 0$, so that $\mathbf{\Gamma}_R$ vanishes.

If the fins are not too small, nearly all the damping couple is due to the fact that, because of the lateral velocity ρr of each part of the fins, the true angle of incidence (downwash being neglected) is not Δ_ρ but

$$\Delta_\rho - \rho r/V = \rho\left(\frac{1}{P} - \frac{r}{V}\right) \tag{3·53·5}$$

by Assumption A 1. At the centre of pressure of a fin this is

$$\Delta_F - a_F r/V.$$

Hence in this case we have

$$\Gamma_A a = \Gamma_F a_F, \tag{3·53·6}$$

approximately. For the purpose of developing the theory, however, we only require (3·53·3, 4) and need make no assumptions regarding the coefficients Γ_A and Γ_F at this stage.

Drum fins. In many cases the fins are wholly or partly enclosed by a cylindrical drum coaxial with the axis of the projectile. Formulae (3·53·4, 6) can be used in this case also, although the values of Γ_A, Γ_F and a_F will be different, in general.

The effect of wind. We saw in § 3·52 that the effect of the wind on an ideal shell was to introduce an extra lateral force $\mathbf{L}'_I$ and couple $\mathbf{M}'_I$. The change in this force and couple due to offsetting the fins is of the second order, and so is the effect of the wind upon the axial torque $\mathbf{\Gamma}_R$. It follows that the extra forces and couples due to the wind can be taken to be the same as those which act upon the associated ideal shell (i.e. the same shell with straight fins).

Summary. The presence of offset fins introduces a couple $\mathbf{\Gamma}_R$ defined by (3·53·4). The other forces and couples are the same as those which act upon the projectile when fitted with similar straight fins.

3·54. *Asymmetrical projectile*

We consider here the forces acting upon the rocket's equivalent shell when the form of this shell departs slightly from that of the symmetric shells considered in §§ 3·51 to 3·53. The conclusion we obtain is that, provided that the departures are of the first order, their effects upon the force and couple system are negligible to the order of approximation to which we are working under Assumptions A 1 and A 9 (see § 3·55).

The reader may take this for granted if he wishes and omit the remainder of this section, which defines what is meant by first-order departures and justifies the previous statement.

We return to the notation of § 3·511 and denote by F_1, F_2, G_1 and G_2 the lateral force and couple components as before. Dashes denote values applying to the asymmetrical equivalent shell. For an ideal shell or one such as considered in § 3·53, we have, for motion in still air,

$$F^* = f_1 V^* + f_2 \omega^*, \quad G^* = g_1 V^* + g_2 \omega^*, \tag{3·54·1}$$

as in (3·511·1, 2), where f_1, f_2, g_1 and g_2 are complex quantities.

Owing to the asymmetry of the equivalent shell a representation of this type in terms of eight real coefficients is not possible, and sixteen coefficients are required. We may accordingly write

$$F^{*\prime} = f_1' V^* + f_2' \omega^* + f_3' \bar{V}^* + f_4' \bar{\omega}^*, \quad G^{*\prime} = g_1' V^* + g_2' \omega^* + g_3' \bar{V}^* + g_4' \bar{\omega}^*, \tag{3·54·2}$$

where the f_i', g_i' $(i = 1, 2, 3, 4)$ are complex quantities and the bar denotes a conjugate value.

Similarly, we have

$$\left.\begin{aligned} \mathbf{R} &= -R_A V^2 \mathbf{OC}_0, & \mathbf{R}' &= -R_A' V^2 \mathbf{OC}_0, \\ \mathbf{\Gamma} &= -\Gamma_A a V_r \mathbf{OC}_0, & \mathbf{\Gamma}' &= -\Gamma_A' a V_r \mathbf{OC}_0, \\ \mathbf{\Gamma}_R &= \Gamma_F V^2 \mathbf{OC}_0, & \mathbf{\Gamma}_R' &= \Gamma_F' V^2 \mathbf{OC}_0. \end{aligned}\right\} \tag{3·54·3}$$

We make the following assumption:

A 7. *There exists a symmetrical shell of the type considered in* §§ 3·52 *or* 3·53 *such that departures of the equivalent shell from the form of this shell are of the first order, i.e.*

$|f_1' - f_1|$, $|f_3'|$ *are small in comparison with* $|f_1|$,
$|f_2' - f_2|$, $|f_4'|$ *are small in comparison with* $|f_2|$,
$|g_1' - g_1|$, $|g_3'|$ *are small in comparison with* $|g_1|$,
$|g_2' - g_2|$, $|g_4'|$ *are small in comparison with* $|g_2|$,

$|R_A' - R_A|$ *is small in comparison with* R_A,
$|\Gamma_A' - \Gamma_A|$ *is small in comparison with* Γ_A,
$|\Gamma_F' - \Gamma_F|$ *is small in comparison with* Γ_F.

These conditions are not, of course, likely to be of much value as a practical criterion.

As a consequence of Assumptions A 6 and 7, it follows that the replacement of F_1', F_2', G_1', G_2', Γ_A' and Γ_F' by the values F_1, F_2, G_1, G_2, Γ_A and Γ_F appropriate to the symmetrical shell introduces second-order terms which are negligible by Assumption A 1. The substitution of R_A for R_A' can also be neglected according to our convention, since **R** is small in comparison with the axial component of the thrust during burning. See Assumption A 9 in § 3·55, and the remarks at the end of § 3·518.

It has been assumed here that the motion takes place in still air. This has been done for simplicity and clearly does not introduce any fresh considerations provided that we regard V as the velocity relative to the air in (3·54·1 to 3·54·3).

DEFINITION 4. *The symmetric shell from which the actual equivalent shell differs is called the associated symmetrical shell.*

It follows that the associated symmetrical shell is either (*a*) an ideal shell as defined by Definition 3, or (*b*) a shell with offset fins of the form described in § 3·53.

3·55. *The aerodynamic forces on the exit plane*

We consider here the force $\mathbf{L}_e$ and the couple $\mathbf{M}_e$ which were defined in (3·5·2). The following two assumptions are made. The first is analogous to that made in § 2·5 concerning the variation of ρv_N.

A 8. *Variations in the pressure* $\mathbf{p}$ *across the exit plane may be neglected.*

Thus we have, since Σ_0 is the area of the exit plane,

$$\mathbf{L}_e = p\Sigma_0 \mathbf{OK}, \quad \mathbf{M}_e = p\Sigma_0 \mathbf{R}_N \times \mathbf{OK} = -p\Sigma_0 l \mathbf{ON}' \times \mathbf{OK}. \tag{3·55·1}$$

The second assumption has already been partly used in the preceding sections.

A 9. *During the burning period*

$$R, \quad p\Sigma_0 \quad \textit{and} \quad mg\cos\alpha$$

are small in comparison with the thrust QW.

Accordingly, by Assumptions A 1 and A 8, we have

$$\mathbf{L}_e . \mathbf{OZ} = \mathbf{L}_e . \mathbf{OC}_0 = p\Sigma_0, \tag{3·55·2}$$

$$\mathbf{M}_e . \mathbf{OZ} = \mathbf{M}_e . \mathbf{OC}_0 = 0, \tag{3·55·3}$$

$$\mathbf{L}_e . (\mathbf{OX} + i\mathbf{OY}) = p\Sigma_0 (\alpha_K e^{i(\sigma+\phi_K)} + \zeta), \tag{3·55·4}$$

and

$$\mathbf{M}_e . (\mathbf{OX} + i\mathbf{OY}) = ip\Sigma_0 l(\alpha_N e^{i(\sigma+\phi_N)} - \alpha_K e^{i(\sigma+\phi_K)}). \tag{3·55·5}$$

Because of Assumption A 9 and the remarks on orders of magnitude at the end of § 3·518, the right-hand sides of equation (3·55·4, 5) are second-order terms and may be neglected, i.e.

$$\mathbf{L}_e . (\mathbf{OX} + i\mathbf{OY}) = 0, \tag{3·55·6}$$

and

$$\mathbf{M}_e . (\mathbf{OX} + i\mathbf{OY}) = 0. \tag{3·55·7}$$

3·56. *Summary of the aerodynamic forces and couples*

By Assumptions A 1 to 9, the aerodynamic forces and couples acting upon a rocket during burning are:

(i) The force $\mathbf{L}_e$ and $\mathbf{M}_e$ defined by (3·5·2). The components of $\mathbf{L}_e$ and $\mathbf{M}_e$ are given by (3·55·2 to 3·55·7).

(ii) The force $\mathbf{L}_S$ and couple $\mathbf{M}_S$ acting upon the equivalent shell. These may be taken to be the force and couple acting upon the associated symmetrical shell, and consist of the following:

(*a*) A force which is the sum of the forces

$$\mathbf{L}_I = \mathbf{R} + \mathbf{L}_1 + \mathbf{L}_2 + \mathbf{L}_3 + \mathbf{L}_4 \quad \text{(still air)},$$
$$\mathbf{L}'_I = \mathbf{L}'_1 + \mathbf{L}'_3 \quad \text{(wind)}.$$

(*b*) A couple about the centre of gravity which is the sum of the couples

$$\mathbf{M}_I = \mathbf{\Gamma} + \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 + \mathbf{M}_4 \quad \text{(still air)},$$
$$\mathbf{M}'_I = \mathbf{M}'_1 + \mathbf{M}'_3 \quad \text{(wind)},$$
$$\mathbf{\Gamma}_R \quad \text{(offset fins)}.$$

The components of these forces and couples along $\mathbf{OX}$, $\mathbf{OY}$, $\mathbf{OZ}$ and $\mathbf{OA}_0$, $\mathbf{OB}_0$, $\mathbf{OC}_0$ are given in equations (3·517·3 to 3·517·8), (3·52·5 to 3·52·12) and (3·53·4).

Assumption A 7 restricts the form of the external surface of the projectile to such forms as are approximately those of ideal shells except that the fins may be offset each by an equal amount. Thus rockets possessing symmetry of order 2—e.g. projectiles with two fins or projectiles driven by two cylindrical motors placed side by side—do not come under consideration. It is necessary to make these exclusions when three-dimensional motion is considered, since the behaviour of such a projectile varies according to its initial orientation. When the motion is purely two-dimensional, however, the theory can easily be made to apply to projectiles of this type.

3·6. Assumptions

The assumptions made during the course of the investigation fall into three groups.

Assumptions A are made at the outset and are necessary if any progress is to be made. They are sufficiently general to enable the theory to deal with all types of rocket motion (using solid fuels) which can be foreseen at present. At the same time they exclude unnecessary complications which are not vital to the investigation.

Assumptions B consist of reasonable assumptions (for the most part relating to the aerodynamic coefficients) by means of which the equations of motion can be solved mathematically in explicit, though possibly complicated, forms.

Assumptions C consist of more drastic assumptions which may not hold for several types of motion, but which cause considerable simplifications in the mathematics, and lead to solutions which may conveniently be used in routine numerical computations. Even where the assumptions are not strictly valid, these solutions may have a qualitative if not an accurately quantitative value.

3·61. *Assumptions A*

These consist of the assumptions made in § 2, in particular the more specialized assumptions of § 2·5, Assumptions A 1 to 9 of §§ 3·3 and 3·5, and the following six assumptions.

A 10. *The following forces are of the second order (in comparison with the thrust QW) and can be neglected:*

$$2\frac{Q^2 l}{m}, \quad 2\frac{Q^2 q_1}{m}, \quad \frac{dQ}{dt}l.$$

A 11. *The two transverse moments of inertia A and B are approximately equal, i.e. $|A-B|$ is small in comparison with A.*

A 12. *q_2 is small in comparison with l.*

The distance q_2 is defined by (2·4·10). The assumption may be justified by the following rough argument which applies to an unrotated rocket. It can be modified to make it apply to rotated rockets.

By (2·5·4)

$$\int_B \mathbf{v}\rho\, d\tau = -Qq_1\mathbf{n},$$

where $\mathbf{n}$ is the unit vector $\mathbf{OG}$. Suppose that $\mathscr{C}$ is the volume occupied by the burning gases in the rocket. Except for those parts of $\mathscr{C}$ which are contained in a thin layer next to parts of the boundary (the burning surface of the charge and the forward end of the combustion chamber) the gas velocity $\mathbf{v}$ is approximately along the rocket's axis. Let $\mathscr{C}_1$ be this region. Then in $\mathscr{C}_1$ we have

$$|\mathbf{v}\times\mathbf{n}| \leqslant v\epsilon,$$

where ϵ is small. The mass of the gases in $\mathscr{C}-\mathscr{C}_1$ is small in comparison with the total mass of gas in $\mathscr{C}$ and its velocity is low in comparison, so that we have

$$\int_{\mathscr{C}} \mathbf{R}\times\mathbf{v}\rho\,d\tau = \int_{\mathscr{C}_1} \mathbf{R}\times\mathbf{v}\rho\,d\tau \quad \text{and} \quad \int_{\mathscr{C}} \mathbf{v}\rho\,d\tau = \int_{\mathscr{C}_1} \mathbf{v}\rho\,d\tau,$$

approximately. For convenience we measure the vector $\mathbf{R}$ from a fixed point H half-way between the head of the combustion chamber and N the centre of the exit plane, and assume that the total distance between these points is less than $2l$ (i.e. the centre of gravity lies forward of H). Then $|R|\leqslant l$ in $\mathscr{C}$ so that we have, from (2·4·10),

$$\begin{aligned} Qq_1q_2 &= -\mathbf{n}.\int_{\mathscr{C}}(\mathbf{R}\times\mathbf{v})\,\rho\,d\tau \doteqdot -\mathbf{n}.\int_{\mathscr{C}_1}(\mathbf{R}\times\mathbf{v})\,\rho\,d\tau \\ &= \int_{\mathscr{C}_1}\mathbf{R}.(\mathbf{n}\times\mathbf{v})\,\rho\,d\tau \leqslant l\epsilon\int_{\mathscr{C}_1} v\rho\,d\tau \\ &\doteqdot l\epsilon\left|\int_{\mathscr{C}_1}\mathbf{v}\rho\,d\tau\right| \doteqdot l\epsilon\left|\int_{\mathscr{C}}\mathbf{v}\rho\,d\tau\right| \\ &= Qq_1 l\epsilon. \end{aligned}$$

Hence $q_2\leqslant l\epsilon$.

A 13. *The following quantities are small in comparison with unity:*

$$\frac{k'}{m},\quad \frac{k'_M r}{mV},\quad \frac{2Ql}{mV}.$$

This is true for all rockets so far investigated. The assumption is, however, not absolutely necessary in the mathematical development, and its use can be avoided by modifying the definition of the quantities λ, μ_2, μ_3 and E which are introduced in § 4·2. See footnote to § 4·3.

A 14. *Quantities which are negligible in comparison with others with which they are combined remain negligible after differentiation with respect to the time, i.e. second-order quantities remain of the second order after differentiation.*

Thus, if we have an equation of the form

$$\frac{d}{dt}(T_1+T_2) = T_3,$$

where T_2 is of the second order of smallness, then we may approximate by writing

$$\frac{d}{dt}T_1 = T_3,$$

the error being of the second order.

Assumptions of this kind are common in approximate mathematical work of this nature and are often made without explicit mention. It is convenient to have the assumption stated in order to avoid detailed investigations into the justification of every differentiation made. In most cases where it is used it can be justified by means of Assumptions A 1, 9 and 11. It could, however, be justified in all cases by addition to the lists of quantities assumed to be small.

A 15. *With the exception of the burning gases, the configuration of the component parts of the rocket is fixed, i.e. there is no motion of the components relative to the rocket's surface* Σ.

This last assumption excludes, for example, (i) rockets driven by liquid fuel, (ii) movement of the sticks of propellant relative to the rocket's motor, (iii) rockets carrying fins mounted on sleeves which can rotate about the rocket's body, and (iv) rockets carrying liquid-filled shell.

3·62. *Assumptions B*

These assumptions, together with the Assumptions A, are used in § 5 and concern the quantities defined in § 4·2. The symbol γ is defined by (5·1·1) as the ratio of the spin to the velocity and p^2 is given by (5·1·4).

B 1. *The following quantities are small in comparison with* $V\,|\,p^2\,|$:

$$2\kappa\alpha_1,\quad d\alpha_1/dt,\quad 2\kappa\beta_1\gamma,\quad \gamma(d\beta_1/dt).$$

B 2. *The following quantities are small in comparison with* $V^2\,|\,p\,|^2$:

$$\kappa^2,\quad \kappa f/V,\quad d\kappa/dt.$$

B 3. *The quantities* n_1^2, β_1 *and* ϖ_1 *are constant during burning.*

B 4. *The quantities* α_1 *and* κ *are constant during burning.*

The last assumption is not required in § 5 although its introduction permits certain simplifications.

Except when the stability is approximately 'neutral' (i.e. $n^2 = 0$), $n^2 = kd_1/A$ is large in comparison with α_1^2, so that $n_1^2 = n^2$ (see (4·2·2)). Similarly, it is probable that $\beta-\beta_1$ is small in comparison with β, and that $\varpi-\varpi_1$ is small in comparison with ϖ provided that the centre of gravity and the centre of pressure of the Magnus couple $\mathbf{M}_3$ are not too close together.

Accordingly, except in these special cases, Assumption B 3 may be replaced by the equivalent assumptions respecting n^2, β and ϖ. These quantities depend upon the aerodynamic coefficients and upon the two moments of inertia A and C, and may be expected to remain approximately constant at subsonic velocities, provided that the variation in total mass is not too large. The subsonic values of the quantities n^2 and ϖ may be used at higher velocities without introducing too great an error, provided that the velocity remains subsonic for a sufficient period during the initial stages of flight. A more detailed discussion of this is given in § 6·1. See also § 3·518.

3·63. *Assumptions C*

These assumptions are made in § 6 in order to simplify the solution of the equations as far as possible. They refer to quantities defined in §§ 4 and 5, and are additional to the Assumptions A and B.

C 1. *The acceleration is constant between launch and burnt.*

It is not necessary to assume that f is constant while the projectile is on the projector, and this will in fact be the period when the greatest variation in f occurs.

C 2. *The following quantities may be neglected:*

$$\kappa,\quad \alpha_1,\quad \alpha_2,\quad \beta-\beta_1,\quad \beta-\beta_2,\quad \varpi.$$

It follows from Assumptions B 3 and C 2 that n^2 $(=-\nu^2)$ and β are constant, and that ϖ_1 may be neglected.

For comments on the validity of C 1 and 2 see § 6·1.

In addition to Assumptions C 1 and 2, Assumption C 3 is made in §§ 6·9 and 6·10 regarding the variation during burning of the different terms constituting the 'tolerance functions' μ_1, μ_2 and μ_3 of (4·2·9, 10, 11). We may write each of μ_1, μ_2, μ_3 in the form

$$\mu_\nu = \sum_P \{a_P(\nu) + irb_P(\nu)\} \alpha_P e^{i\phi_P}, \tag{3·63·1}$$

where $a_P(\nu)$ and $b_P(\nu)$ are coefficients which are constant or vary slowly during burning (see § 6·10). The assumption is as follows:

C 3. *The following quantities are constant during burning:*

$$a_P(\nu)\,\alpha_P, \quad b_P(\nu)\,\alpha_P, \quad \phi_P \quad (P = C, G, L, M, N, R).$$

In actual fact the quantities α_P and ϕ_P will probably vary in an irregular manner during burning, but Assumption C 3 is justifiable in order to obtain an indication of the relative maximum magnitudes of the effects of the different asymmetries. See §§ 6·9, 6·10 for further discussion of this assumption.

3·7. Definitions

A large number of definitions has already been made, in particular Definitions 1 to 4 of § 3·5. We collect here a number of definitions which are frequently used in the succeeding work. Other definitions are introduced more conveniently as the need arises.

The instant of ignition is the instant at which the projectile first commences to move relative to the projector.

The instant of launch is the last instant at which the projectile is in contact with any part of the projector.

Burnt is the instant at which burning of the propellant ceases.

The direction of projection is that of the fixed vector **OZ** in figure 3. The direction of this vector will usually be that of the axis of the projector, i.e. the direction in which the projectile first starts to move. This direction makes an angle called the *quadrant elevation* (Q.E.) with the horizontal plane.

The line of projection is the line through the centre of gravity at the instant $t = t_0$ parallel to **OZ**.

The line of fire is the projection of the line of projection on the horizontal plane.

The plane of fire is the plane through the line of projection which is normal to the vertical plane through the lines of projection and fire.

The rocket's axis, the angular deviation of the trajectory from the direction of projection, and the yaw have already been defined (§§ 3·5, 3·211, 3·213).

In order to calculate the trajectory (i.e. the path of the centre of gravity) and motion of the projectile about its centre of gravity after launch, it is necessary to know the following quantities† at launch:

$$V_0, \quad r_0; \qquad Z_0, \quad \Xi_0, \quad \zeta_{01} = \left(\frac{d\zeta}{dt}\right)_{t=t_0}. \tag{3·7·1}$$

The first two are usually assumed to be known without requiring mention. The values of the remaining three complex quantities Z, Ξ and $d\zeta/dt$ at launch are called the *initial*

† Since Z_0, Ξ_0 and ζ_{01} are complex, eight real quantities in all are required.

conditions. They specify the magnitudes and orientations of the initial angular deviation, yaw and cross-spin.

A rocket is said to be *perfectly launched* when, at the instant of launch,

$$Z_0 = \Xi_0 = 0, \quad \zeta_{01} = 0. \tag{3·7·2}$$

In practice, perfect launch is unlikely to occur. Its value as a concept is that, owing to the linearity of the equations of motion, the behaviour of the projectile may be regarded as being the sum of the four contributions due to the motions resulting from (i) perfect launch, (ii) an initial angular deviation Z_0, (iii) an initial yaw Ξ_0, and (iv) an initial rate of turn ζ_{01}. The last three are usually referred to as *the effect of tip-off*, since the initial conditions will normally be due to the tilting of the projectile over the end of the projector before launch occurs. See also § 4·4.

4. Formation and reduction of the general equations of motion

4·1. General vector form and resolution

Under Assumptions A, the general equations of motion of the rocket after launch are given in vector form by (2·5·11, 12), where

$$\mathbf{L} = \mathbf{L}_e + \mathbf{L}_I + \mathbf{L}'_I + m\mathbf{g},$$

and

$$\mathbf{M} = \mathbf{M}_e + \mathbf{M}_I + \mathbf{M}'_I + \mathbf{\Gamma}_R,$$

by § 3·56.

We now proceed to multiply these equations scalarly by the vectors $\mathbf{OZ}$ and $\mathbf{OX}+i\mathbf{OY}$ in order to bring them into forms suitable for solution.

By § 3·32 and Assumption A 14, we have for the linear momentum

$$\mathbf{OZ}\,.\,m\frac{d}{dt}\mathbf{V} = m\frac{d}{dt}(V\mathbf{OT}\,.\,\mathbf{OZ}) = m\frac{dV}{dt},$$

and

$$(\mathbf{OX}+i\mathbf{OY})\,.\,m\frac{d}{dt}\mathbf{V} = m\frac{d}{dt}(VZ).$$

The components of the aerodynamic forces are given by equations (3·517·3, 7), (3·52·7, 11) and (3·55·2, 6). The components of the force of gravity and of the 'jet forces' on the right of (2·5·11) can be obtained from the tables in § 3·52. Accordingly, we have

$$\begin{aligned} m\frac{dV}{dt} &= QW + p\Sigma_0 - R - mg\sin\alpha + 2\frac{Q^2}{m}(l-q_1) + l\frac{dQ}{dt} \\ &= QW - (R + mg\sin\alpha - p\Sigma_0) \end{aligned} \tag{4·1·1}$$

by Assumption A 10. In a similar way, and by virtue of the remarks on magnitudes in § 3·518,

$$\begin{aligned} m\frac{d}{dt}(VZ) = {} & (kV - ik_M r)(V\Xi + w_1) + (k'V - ik'_M r + 2Ql)\frac{d\zeta}{dt} \\ & + mg\cos\alpha + QW\zeta + QW e^{i\sigma}\left\{\alpha_R e^{i\phi_R} + 2i\frac{rl}{W}\alpha_N e^{i\phi_N}\right\}. \end{aligned} \tag{4·1·2}$$

Hence, by (4·1·1) and Assumption A 9,

$$mV\frac{dZ}{dt} = QW\Xi + (kV - ik_M r)(V\Xi + w_1) + (k'V - ik'_M r + 2Ql)\frac{d\zeta}{dt}$$
$$+ mg\cos\alpha + QW e^{i\sigma}\left\{\alpha_R e^{i\phi_R} + 2i\frac{rl}{W}\alpha_N e^{i\phi_N}\right\}. \qquad (4·1·3)$$

The angular momentum $\mathbf{h}_G$ is given by

$$\mathbf{h}_G = A\Omega_a\mathbf{OA} + B\Omega_b\mathbf{OB} + C\Omega_c\mathbf{OC} = A\mathbf{\Omega} + (C-A)\Omega_c\mathbf{OC} - (A-B)\Omega_b\mathbf{OB}. \qquad (4·1·4)$$

Hence, by Assumptions A 11 and 14,

$$\mathbf{OZ}.\frac{d}{dt}\mathbf{h}_G = \frac{d}{dt}(\mathbf{OZ}.\mathbf{h}_G)$$
$$= \frac{d}{dt}\{Ar + (C-A)r - (A-B)\Omega_b(\alpha_C\sin\psi_C + \theta\sin(\phi+\sigma_C))\}$$
$$= \frac{d}{dt}(Cr), \qquad (4·1·5)$$

and $$(\mathbf{OX}+i\mathbf{OY}).\frac{d}{dt}\mathbf{h}_G = \frac{d}{dt}\{(\mathbf{OX}+i\mathbf{OY}).\mathbf{h}_G\}$$
$$= \frac{d}{dt}\left\{iA\frac{d\zeta}{dt} + Ar\zeta + (C-A)r(\zeta + \alpha_C e^{i(\sigma+\phi_C)}) - i(A-B)\Omega_b e^{i(\sigma+\sigma_C)}\right\}$$
$$= \frac{d}{dt}\left\{iA\frac{d\zeta}{dt} + Cr\zeta + (C-A)r\alpha_C e^{i(\sigma+\phi_C)}\right\}. \qquad (4·1·6)$$

The components of the aerodynamic couples are given by the equations (3·517·4, 8), (3·52·8, 12), (3·53·4) and (3·55·3, 7). The components of the jet couples on the right of (2·5·12) can be obtained from the tables in § 3·32. Accordingly we have, by (4·1·5, 6),

$$\frac{d}{dt}(Cr) = G_R + \Gamma_F V^2\Delta_F - \Gamma_A Var - Qk_e^2 r, \qquad (4·1·7)$$

and $$\frac{d}{dt}\left\{iA\frac{d\zeta}{dt} + Cr\zeta + (C-A)r\alpha_C e^{i(\sigma+\phi_C)}\right\} = \zeta\{G_R + \Gamma_F V^2\Delta_F - \Gamma_A Var - Qk_e^2 r\}$$
$$-(ikd_1 V + k_M d_3 r)(V\Xi + w_1) + (-ik'd_2 V - k'_M d_4 r - iQl^2 - \tfrac{1}{2}iQk_e^2 + Qq_1 q_2)d\zeta/dt$$
$$+ e^{i\sigma}\{G_R\alpha_L e^{i\phi_L} + iQWl(\alpha_N e^{i\phi_N} - \alpha_R e^{i\phi_R}) + Ql^2 r\alpha_N e^{i\phi_N}$$
$$-\tfrac{1}{2}Qk_e^2 r\alpha_K e^{i\phi_K} + q_1 l[i(dQ/dt) - Qr](\alpha_M e^{i\phi_M} - \alpha_G e^{i\phi_G}) + Qq_1 q_2 ir\alpha_G e^{i\phi_G}\}. \qquad (4·1·8)$$

Hence, by (4·1·7, 8) and Assumptions A 1, 9 and 12,

$$A\frac{d^2\zeta}{dt^2} + \frac{d\zeta}{dt}\left\{\frac{dA}{dt} + Ql^2 - iCr + k'd_2 V - ik'_M d_4 r\right\} + (kd_1 V - ik_M d_3 r)(V\Xi + w_1)$$
$$= e^{i\sigma}\{-iG_R\alpha_L e^{i\phi_L} + Ql(W - ilr)\alpha_N e^{i\phi_N} - QlW\alpha_R e^{i\phi} - iQq_1 lr\alpha_G e^{i\phi_G}$$
$$+ iQq_1 lr\alpha_M e^{i\phi_M}\} - i\frac{d}{dt}\{(A-C)r\alpha_C e^{i(\sigma+\phi_C)}\}. \qquad (4·1·9)$$

The motion of the projectile is completely determined by equations (4·1·1, 3, 7, 9).

4·2. Further notation

It is convenient, at this point, to introduce the following notation. Write

$$f_Q = \frac{QW}{m}, \quad f_R = \frac{1}{m}(R + mg\sin\alpha - p\Sigma_0), \tag{4·2·1}$$

$$n^2 = -\nu^2 = \frac{kd_1}{A}, \quad n_1^2 = n^2 - \alpha_1^2, \tag{4·2·2}$$

$$\varpi = \frac{k_M d_3}{A}, \quad \varpi_1 = \frac{k_M d_3}{A} + 2\alpha_1\beta_1, \tag{4·2·3}$$

where

$$2\alpha_1 = \frac{k}{m} - \frac{k' d_2}{A}, \quad 2\alpha_2 = \frac{k}{m} + \frac{k' d_2}{A}, \tag{4·2·4}$$

$$2\beta_1 = \frac{C}{A} + \frac{k'_M d_4}{A} - \frac{k_M}{m}, \qquad 2\beta_2 = \frac{C}{A} + \frac{k'_M d_4}{A} + \frac{k_M}{m}, \tag{4·2·5}$$

$$\beta = \frac{C}{2A}, \tag{4·2·6}$$

$$2A\kappa = Ql^2 + dA/dt, \tag{4·2·7}$$

$$A\lambda = 2A\kappa + k' d_2 V - ir(C + k'_M d_4), \tag{4·2·8}$$

$$\mu_1 = \alpha_R e^{i\phi_R} + 2i\frac{lr}{W}\alpha_N e^{i\phi_N}, \tag{4·2·9}$$

$$\mu_2 = \frac{ml}{A}\left\{\left(1 - i\frac{lr}{W}\right)\alpha_N e^{i\phi_N} - \alpha_R e^{i\phi_R} - i\frac{G_R}{QWl}\alpha_L e^{i\phi_L}\right.$$
$$\left. - i\frac{q_1 r}{W}\alpha_G e^{i\phi_G} + i\frac{q_1 r}{W}\alpha_M e^{i\phi_M} - i\frac{(1-2\beta)}{QWl}\frac{dA}{dt} r\alpha_C e^{i\phi_C}\right\}, \tag{4·2·10}$$

$$\mu_3 = -i(1 - 2\beta)\, r\alpha_C e^{i\phi_C}, \tag{4·2·11}$$

$$E = \frac{V}{A}(kd_1 V - ik_M d_3 r) = n^2 V^2 - i\varpi V r, \tag{4·2·12}$$

$$F = \frac{1}{m}(kV - ik_M r), \tag{4·2·13}$$

$$F_1 = \frac{1}{mV}(k' V - ik'_M r + 2Ql), \tag{4·2·14}$$

$$P(s) = \frac{1}{2}\int_{s_0}^{s} \frac{F+\lambda}{V}\, ds = \int_{t_0}^{t} (\kappa + \alpha_2 V - i\beta_2 r)\, dt, \tag{4·2·15}$$

$$\Lambda = \Lambda(s) = \frac{F-\lambda}{2V} = \frac{1}{V}(-\kappa + \alpha_1 V + i\beta_1 r), \tag{4·2·16}$$

$$G(s) = \frac{1}{V^2}\left\{E + \left(\lambda - \frac{f}{V}\right)F + \frac{dF}{dt}\right\} - \{P'(s)\}^2 - P''(s)$$
$$= n^2 - i\varpi\frac{r}{V} - \Lambda^2(s) + \Lambda'(s)$$
$$= \frac{1}{V^2}\{G_1(s) - irG_2(s)\}, \tag{4·2·17}$$

where $$G_1(s) = n_1^2 V^2 + \beta_1^2 r^2 + 2\kappa\alpha_1 V + \kappa\frac{f}{V} - \kappa^2 - \frac{d\kappa}{dt} + V\frac{d\alpha_1}{dt}, \tag{4·2·18}$$

and $$G_2(s) = \varpi_1 V + \beta_1 \frac{f}{V} - 2\kappa\beta_1 - \frac{d\beta_1}{dt} - \frac{\beta_1}{r}\frac{dr}{dt}, \tag{4·2·19}$$

$$T(s) = T_1(s) + T_2'(s), \tag{4·2·20}$$

where $$T_1(s) = e^{i\sigma}\frac{f}{V^2}(\lambda\mu_1 - \mu_2 V), \tag{4·2·21}$$

and $$T_2(s) = e^{i\sigma}\left(\mu_1\frac{f}{V} - \mu_3\right), \tag{4·2·22}$$

$$\mathrm{H} = \mathrm{H}(s) = \eta\, e^{P(s)} = (V\Xi + w_1)\, e^{P(s)}. \tag{4·2·23}$$

4·3. Further reduction of the equations

On the right-hand sides of (4·1·3, 9) we may replace QW by mf wherever it occurs, since, by (4·1·1) and Assumption A 9, the error introduced is of the second order. Thus we have, in the notation of § 4·2,

$$\boxed{\frac{dV}{dt} = f = f_Q - f_R}, \tag{4·3·1}$$

$$\boxed{C\frac{dr}{dt} + r\left(\frac{dC}{dt} + \Gamma_A V_a + Qk_e^2\right) = G_R + \Gamma_F V^2 \Delta_F}, \tag{4·3·2}$$

$$\frac{d\mathrm{Z}}{dt} = \frac{f}{V}(\Xi + \mu_1 e^{i\sigma}) + F\left(\Xi + \frac{w_1}{V}\right) + F_1\frac{d\zeta}{dt} + \frac{g}{V}\cos\alpha, \tag{4·3·3}$$

and $$\frac{d^2\zeta}{dt^2} + \lambda\frac{d\zeta}{dt} + E\left(\Xi + \frac{w_1}{V}\right) = f\mu_2 e^{i\sigma} + \frac{d}{dt}(\mu_3 e^{i\sigma}). \tag{4·3·4}$$

By Assumptions A 1 and 13, $F_1\, d\zeta/dt$ is a second-order quantity and may accordingly be neglected† in equation (4·3·3). If we eliminate Z and ζ from this equation and (4·3·4), using the relation $\Xi = \zeta - \mathrm{Z}$, we obtain

$$\frac{d^2\Xi}{dt^2} + \frac{d\Xi}{dt}\left(\lambda + F + \frac{f}{V}\right) + \Xi\left\{E + \lambda\left(F + \frac{f}{V}\right) + \frac{dF}{dt} + \frac{d}{dt}\left(\frac{f}{V}\right)\right\}$$
$$= -\frac{g\cos\alpha}{V}\left(\lambda - \frac{f}{V}\right) - \frac{w_1}{V}\left\{E + F\left(\lambda - \frac{f}{V} + \frac{\dot{w}_1}{w_1}\right) + \frac{dF}{dt}\right\} - VT(s). \tag{4·3·5}$$

We now change the dependent variable from Ξ to $V\Xi$ and the independent variable from t to s. Then (4·3·5) becomes

$$\frac{d^2}{ds^2}(V\Xi) + \frac{\lambda + F}{V}\frac{d}{ds}(V\Xi) + \frac{1}{V^2}\left\{E + F\left(\lambda - \frac{f}{V}\right) + \frac{dF}{dt}\right\}V\Xi$$
$$= -\frac{g\cos\alpha}{V^2}\left(\lambda - \frac{f}{V}\right) - \frac{w_1}{V^2}\left\{E + F\left(\lambda - \frac{f}{V} + \frac{\dot{w}_1}{w_1}\right) + \frac{dF}{dt}\right\} - T(s). \tag{4·3·6}$$

† If the term $F_1\, d\zeta/dt$ is retained in (4·3·3), the equations (4·3·5 to 4·3·9) hold with $\lambda + \frac{dF_1/dt}{1-F_1}$ in place of λ, and E, μ_2 and $\frac{d}{dt}(\mu_3 e^{i\sigma})$ multiplied by $1 - F_1$ wherever they occur. This amounts to a modification of the definitions of λ, μ_2, μ_3 and E and of the quantities depending upon them.

Hence, by (4·2·23),

$$\frac{d^2\eta}{ds^2}+\frac{\lambda+F}{V}\frac{d\eta}{ds}+\frac{1}{V^2}\left\{E+F\left(\lambda-\frac{f}{V}\right)+\frac{dF}{dt}\right\}\eta$$

$$=\left\{w_1''(s)+\frac{\lambda}{V}w_1'(s)\right\}-\frac{g\cos\alpha}{V^2}\left(\lambda-\frac{f}{V}\right)-T(s), \tag{4·3·7}$$

and therefore

$$\boxed{\frac{d^2\mathrm{H}}{ds^2}+G(s)\,\mathrm{H}=\mathrm{e}^{P(s)}\left\{w_1''(s)+\frac{\lambda}{V}w_1'(s)-\frac{g\cos\alpha}{V^2}\left(\lambda-\frac{f}{V}\right)-T(s)\right\}}. \tag{4·3·8}$$

In terms of η we obtain, by (4·3·3),

$$\frac{dZ}{dt}=\frac{1}{V}\frac{d\eta}{dt}-\frac{d\Xi}{dt}+F\frac{\eta}{V}-\frac{\dot{w}_1}{V}+\frac{g}{V}\cos\alpha+\frac{f\mu_1}{V}\mathrm{e}^{i\sigma}.$$

Hence

$$\boxed{\frac{dZ}{ds}=-\frac{d\Xi}{ds}+\frac{\mathrm{e}^{-P(s)}}{V}\{\Lambda(s)\,\mathrm{H}(s)+\mathrm{H}'(s)\}+\frac{1}{V^2}\{g\cos\alpha+f\mu_1\,\mathrm{e}^{i\sigma}-Vw_1'(s)\}.} \tag{4·3·9}$$

Thus the four equations which determine the motion are (4·3·1, 2, 8, 9). It is worth pointing out at this stage that the forces L_2 and L_4 and the quantities α_K and q_2 have been omitted from these equations since they produce second order effects (cf. footnote to p. 497).

4·4. Method of solution of the equations

In order to determine the motion of and about the centre of gravity it is, first of all, necessary to know the velocity V and spin r. The former is obtained from (4·3·1). When f_Q and f_R are known at each stage during burning, V may be obtained by step by step methods of integration. Since, however, f_R is small in comparison with f_Q—especially at low velocities—an approximate value of V may be obtained by neglecting f_R in (4·3·1). Thus we obtain

$$\frac{dV}{dt}=\frac{QW}{m}=-\frac{W}{m}\frac{dm}{dt}, \tag{4·4·1}$$

and this will hold to the same order of approximation on the projector. Hence

$$V=V_{00}+\int_m^{m_{00}}W\frac{dm}{m} \tag{4·4·2}$$

$$=V_{00}+W\log\frac{m_{00}}{m}, \tag{4·4·3}$$

when W is constant. For ground firing V_{00} is, of course, zero.

Equation (4·3·2) for the axial spin has the explicit solution

$$r=\frac{1}{C}\int_{t_0}^{t}(G_R+\Gamma_F V^2\Delta_F)\exp\left\{-\int_{t'}^{t}(\Gamma_A Va+Qk_e^2)\frac{dt''}{C}\right\}dt'+\frac{C_0}{C}r_0\exp\left\{-\int_{t_0}^{t}(\Gamma_A Va+Qk_e^2)\frac{dt''}{C}\right\}, \tag{4·4·4}$$

where $G_R+\Gamma_F V^2\Delta_F$ is a function of the time t', and $\{\Gamma_A\, Va+Qk_e^2\}/C$ of the time t''.

When no additional rotating couple acts during the interval $(0, t_0)$—or if such a couple is small, e.g. a frictional couple between round the projector—we may use the equation (4·3·2) over the whole range and obtain

$$r = \frac{1}{C}\int_0^t (G_R + \Gamma_F V^2 \Delta_F) \exp\left\{-\int_{t'}^t (\Gamma_A Va + Qk_e^2)\frac{dt''}{C}\right\} dt' + \frac{C_{00}}{C} r_{00} \exp\left\{-\int_0^t (\Gamma_A Va + Qk_e^2)\frac{dt''}{C}\right\}, \quad (4\cdot4\cdot5)$$

which is of a simpler form when, as is usually the case, $r_{00} = 0$.

The velocity and spin being known, (4·3·8) must be solved in order to determine H. When this has been done, the yaw Ξ is known, since

$$\Xi = \frac{1}{V}\{\mathrm{H}\, e^{-P(s)} - w_1\}, \quad (4\cdot4\cdot6)$$

and Z may be obtained from (4·3·9) after performing a single integration when Z_0 is known. It is occasionally useful to know the cross-spin $d\zeta/dt$. This is given by

$$\frac{d\zeta}{dt} = e^{-P(s)}\{\Lambda(s)\,\mathrm{H}(s) + \mathrm{H}'(s)\} + \frac{1}{V}\{g\cos\alpha + f\mu_1 e^{i\sigma} - Vw_1'(s)\}. \quad (4\cdot4\cdot7)$$

The solution of (4·3·8) depends upon the form of the function $G(s)$ and upon the initial conditions. It will be seen from (4·2·17 to 4·2·19) that $G(s)$ is, in general, of a very complicated form, so that explicit mathematical solutions for H are only possible when assumptions have been made as to the relative order of magnitude and rate of variation with s of the various terms which constitute $G(s)$. In particular, the most important factor is the relation between the spin r and the velocity V. If $G(s)$ is a slowly varying function of s with no zeros in the range under consideration, it is possible to obtain approximate solutions of the equation (see § 8). The most elegant and complete solutions are obtained, however, when the axial spin is proportional to the velocity, and this case is investigated in § 5; it includes the case of no axial spin. §§ 6 and 7 contain solutions which are valid under less general assumptions such as constant acceleration.

The general solution of (4·3·8) contains two indeterminate (complex) constants, and one further constant is required for (4·3·9). These three constants will usually be given by the initial conditions at launch (see § 3·7). They are the values of Z, Ξ and $d\zeta/dt$ at this instant, namely Z_0, Ξ_0 and ζ_{01}. The initial values $\mathrm{H}_0 = \mathrm{H}(s_0)$ and $\mathrm{H}_{01} = \mathrm{H}'(s_0)$ are given, in terms of Ξ_0 and ζ_{01}, by the relations

$$\mathrm{H}_0 = V_0 \Xi_0 + w_1(s_0) \quad (4\cdot4\cdot8)$$

and

$$\mathrm{H}_{01} = \zeta_{01} - \Lambda_0 \mathrm{H}_0 - \frac{1}{V_0}\{g\cos\alpha + f_0\mu_1(s_0)\, e^{i\sigma_0} - V_0 w_1'(s_0)\}. \quad (4\cdot4\cdot9)$$

In the following sections the equations (4·3·8, 9) are solved for Z, Ξ and $d\zeta/dt$ under various assumptions regarding the forms of the spin r and acceleration f. There are reasons for obtaining Z, Ξ and $d\zeta/dt$ in preference to other possible variables. If no impulsive forces or couples act, Z, Ξ and $d\zeta/dt$ must be continuous functions of the arc length s (or time t) at each point of the trajectory. Their values at the end of any arc—e.g. the arc described during the burning period—provide the initial conditions determining the subsequent motion. A knowledge of the three quantities is therefore useful when the trajectory consists of a number of arcs on which different conditions prevail. For a rocket the number of such arcs will usually be two, namely, the arc described during burning and the remainder of the trajectory. For multi-stage rockets, however, the trajectory may consist of several arcs.

When the angular deviation Z is known, the linear deviation of the centre of gravity from the line of projection at launch is given by the equation

$$D\,e^{i\Upsilon} = \int_{s_0}^{s} Z\,ds. \tag{4·4·10}$$

Here $D\cos\Upsilon$ is the downward component of the deviation in the vertical plane perpendicular to **OZ**, and $D\sin\Upsilon$ is the linear deviation to the left as viewed from the rear. The height of the centre of gravity above its initial position at launch is, at any instant during burning,

$$(s-s_0)\sin\alpha - D\cos\Upsilon\cos\alpha. \tag{4·4·11}$$

Formulae for $D\,e^{i\Upsilon}$ are not given since they may be obtained from Z by a straightforward integration, and for the following reason. It is during the burning period of a rocket that disturbing factors (apart from gravity), such as wind and the various asymmetries of the projectile, have their greatest effect, and in consequence the trajectory depends, to a considerable extent, upon the direction of motion at the instant of burnt, i.e. upon the angular deviation of the trajectory at burnt. The actual linear displacement at burnt is usually quite insignificant in comparison with the later displacement which is due, primarily, to the angular deviation at burnt.

It will be observed that the right-hand sides of equations (4·3·8, 9) are linear in the eight quantities

$$g; \quad w_1(s); \quad \alpha_C\,e^{i\phi_C}, \quad \alpha_G\,e^{i\phi_G}, \quad \alpha_L\,e^{i\phi_L}, \quad \alpha_M\,e^{i\phi_M}, \quad \alpha_N\,e^{i\phi_N}, \quad \alpha_R\,e^{i\phi_R};$$

and that the values of V and r are independent of them.† These eight quantities correspond to eight 'disturbing factors', namely, gravity, wind and the six 'tolerances' associated with the directions of the principal longitudinal axis of inertia, the charge centre of gravity, the rotational torque axis, the thrust application point, the centre of the exit plane and the thrust. Further, H is linear in w_1, and $G(s)$ is independent of the eight quantities.

It follows that the values of Z, Ξ and $d\zeta/dt$ obtained from (4·3·8, 9) may each be expressed as a complementary function which is a linear function of the three initial values Z_0, Ξ_0 and ζ_{01} plus a particular integral which is a linear combination of eight parts, each part corresponding to a single disturbing factor. For example, the angular deviation may be written in the form

$$Z = \mathscr{Z}_1 Z_0 + \mathscr{Z}_2 \Xi_0 + \mathscr{Z}_3 \zeta_{01} + Z_g + Z_w + Z_C + Z_G + Z_L + Z_M + Z_N + Z_R$$

in an obvious notation. Here, for example, Z_g is the angular deviation due to gravity of a symmetrical rocket when the projectile is perfectly launched and there is no wind.

The three initial values Z_0, Ξ_0 and ζ_{01} will also, in general, be linear functions of the eight disturbing factors, to a similar order of approximation. Thus the total angular deviation Z (and similarly for the yaw and cross spin) may be regarded as a linear sum of eight parts, each of which is due to a single disturbing factor and is made up of four contributions due to (i) an initial deviation, (ii) an initial yaw, (iii) an initial rate of turn, all associated with the disturbing factor in question, and (iv) the action of the disturbing factor after launch when the projectile is perfectly launched.

† Strictly, f and therefore V depends upon g to the first order, by (4·2·1) and (4·3·1); however, in equations (4·3·8, 9), with which we are concerned, we may put $f = f_Q$ which is independent of g, since the error involved is of the second order and can be neglected.

Since the values of Z_0, Ξ_0 and ζ_{01} depend to a considerable extent upon the design of the projectile and of the projector, the general procedure adopted will be to give eleven solutions for each of Z, Ξ and $d\zeta/dt$ corresponding to eleven cases:

(1) Arbitrary initial deviation Z_0; $\Xi_0 = \zeta_{01} = 0$, no disturbing factors.

(2) Arbitrary initial yaw Ξ_0; $Z_0 = \zeta_{01} = 0$, no disturbing factors.

(3) Arbitrary initial rate of turn ζ_{01}; $Z_0 = \Xi_0 = 0$, no disturbing factors.

(4) Gravity; perfect launch and no other disturbing factors.

(5) Wind; perfect launch and no other disturbing factors.

(6) to (11) The six tolerances separately; perfect launch and no other disturbing factors.

The total angular deviation (and similarly for the yaw and rate of turn) due to one or to all of the disturbing factors may then be found from the eleven sets of solutions when the initial values are known, e.g. from experimental determinations or by calculation from the motion on the projector.†

The solution of the first case is, of course, trivial and will be omitted in future; it is

$$Z = Z_0, \quad \Xi = 0, \quad d\zeta/dt = 0. \tag{4·4·12}$$

It has been taken for granted that the forms of the various subsidiary functions such as

$$P(s), \quad \Lambda(s), \quad \lambda, \quad \kappa, \quad T_1(s), \quad T_2(s),$$

etc., occurring in the equations are known at each instant during burning. The quantities upon which these functions depend may be divided into four main groups under the headings: (i) design of projectile (weights, dimensions), (ii) charge, combustion chamber and nozzle characteristics, (iii) aerodynamic force and couple coefficients, and wind structure, and (iv) magnitude and variation of the tolerances. For any particular weapon most will usually be known about the first group and least about the last. Methods of estimating some of these quantities are discussed in § 9.

5. General solution of the equations for a spin proportional to the velocity

5·1. General

We assume throughout this section that the Assumptions A and B 1 to 3 hold (see §§ 3·61, 3·62), and that the axial spin r is related to the velocity in the following way:

$$r = \gamma V. \tag{5·1·1}$$

Here γ is a constant which may be positive, negative or zero. For positive γ this corresponds to a clockwise rotation about the axis when viewed from the rear. The relation (5·1·1) holds to a good degree of accuracy for rockets which are spun by passing the burning gases through inclined nozzles. See, for example, § 9·4. We have, by (4·2·17, 18, 19) and (5·1·1),

$$V^2G(s) = (n_1^2+\beta_1^2\gamma^2-i\gamma\varpi_1)\,V^2+\left\{2\kappa\alpha_1+\frac{d\alpha_1}{dt}+i\gamma\left(2\kappa\beta_1+\frac{d\beta_1}{dt}\right)\right\}V+\kappa\frac{f}{V}-\kappa^2-\frac{d\kappa}{dt}. \tag{5·1·2}$$

By Assumptions B 1, 2 and 3, we may write

$$G(s) = p^2 \tag{5·1·3}$$

† Formulae for Z_0, Ξ_0 and ζ_{01} for two types of projector are given in the monograph mentioned in § 1·2. See also § 9·1.

in (4·3·8), where
$$p^2 = n_1^2+\beta_1^2\gamma^2-i\gamma\varpi_1, \qquad (5\cdot1\cdot4)$$
and is constant. Put
$$p = p_1-ip_2, \qquad (5\cdot1\cdot5)$$
where
$$p_1^2 = \tfrac{1}{2}\{[(n_1^2+\beta_1^2\gamma^2)^2+\varpi_1^2\gamma^2]^{\frac{1}{2}}+n_1^2+\beta_1^2\gamma^2\}, \qquad (5\cdot1\cdot6)$$
and
$$p_2^2 = \tfrac{1}{2}\{[(n_1^2+\beta_1^2\gamma^2)^2+\varpi_1^2\gamma^2]^{\frac{1}{2}}-n_1^2-\beta_1^2\gamma^2\}. \qquad (5\cdot1\cdot7)$$

From (3·3·9) and (4·2·15, 16) we have†
$$\sigma = \gamma(s-s_0)+\sigma_0, \qquad (5\cdot1\cdot8)$$
$$P(s) = \int_{t_0}^{t}\kappa\,dt+\int_{s_0}^{s}(\alpha_2-i\beta_2\gamma)\,ds, \qquad (5\cdot1\cdot9)$$
and
$$\Lambda(s) = -\frac{\kappa}{V}+\alpha_1+i\beta_1\gamma. \qquad (5\cdot1\cdot10)$$

In order to solve the equations it is not necessary to assume more than Assumptions A and B 1 to 3, and we shall not make any further assumptions at this stage. If, however, Assumption B 4 is made, then the following quantities, which occur frequently in the expressions obtained, may be written as
$$\Lambda(u)-\Lambda(v) = -\kappa\left(\frac{1}{V_u}-\frac{1}{V_v}\right), \qquad (5\cdot1\cdot11)$$
and
$$p^2+\Lambda(u)\,\Lambda(v) = n^2-i\gamma\varpi. \qquad (5\cdot1\cdot12)$$

5·2. General solution

Write
$$R(s) = e^{P(s)}\left\{w_1''(s)+\frac{\lambda}{V}w_1'(s)-\frac{g\cos\alpha}{V^2}\left(\lambda-\frac{f}{V}\right)-T(s)\right\}, \qquad (5\cdot2\cdot1)$$
and
$$T_3(s) = -T_1(s)+P'(s)\,T_2(s). \qquad (5\cdot2\cdot2)$$
Then, since
$$\lambda = V\{P'(s)-\Lambda(s)\},$$
we may put
$$R(s) = -R_1(s)+R_2'(s), \qquad (5\cdot2\cdot3)$$
where
$$R_1(s) = e^{P(s)}\left\{\Lambda(s)\left(w_1'(s)-\frac{g\cos\alpha}{V}\right)-T_3(s)\right\}, \qquad (5\cdot2\cdot4)$$
and
$$R_2(s) = e^{P(s)}\left\{w_1'(s)-\frac{g\cos\alpha}{V}-T_2(s)\right\}. \qquad (5\cdot2\cdot5)$$
Also, by (4·2·21, 22),
$$T_3(s) = e^{i\sigma}\left\{\frac{f}{V}(\mu_2+\mu_1\Lambda(s))-\mu_3P'(s)\right\}. \qquad (5\cdot2\cdot6)$$

The general solution of equation (4·3·8) is
$$\mathrm{H}(s) = K_1\cos p(s-s_0)+K_2\sin p(s-s_0)+\frac{1}{p}\int_{s_0}^{s}R(u)\sin p(s-u)\,du, \qquad (5\cdot2\cdot7)$$
where K_1 and K_2 are constants depending upon the initial conditions. By (4·4·8, 9) we have
$$K_1 = V_0\Xi_0+w_1(s_0), \qquad (5\cdot2\cdot8)$$
and
$$K_2 = \frac{1}{p}\left[\zeta_{01}-\Lambda_0V_0\Xi_0-\Lambda_0w_1(s_0)-\frac{1}{V_0}\{g\cos\alpha+f_0\mu_1(s_0)\,e^{i\sigma_0}-w_1'(s_0)\,V_0\}\right]. \qquad (5\cdot2\cdot9)$$

† If (5·1·1) holds in the interval $(0, t_0)$ then $\sigma = \gamma s$, of course.

We may, by (5·2·3), write

$$\mathrm{H}(s) = K_1 \cos p(s-s_0) + K_3 \sin p(s-s_0) + \frac{1}{p}\int_{s_0}^{s} \{pR_2(u)\cos p(s-u) - R_1(u)\sin p(s-u)\}\,du, \tag{5·2·10}$$

where $$K_3 = K_2 - \frac{1}{p}R_2(s_0) = \frac{1}{p}[\zeta_{01} - \Lambda_0 V_0 \Xi_0 - \Lambda_0 w_1(s_0) - \mu_3(s_0)\,\mathrm{e}^{i\sigma_0}]. \tag{5·2·11}$$

It follows that

$$\mathrm{H}'(s) = pK_3 \cos p(s-s_0) - pK_1 \sin p(s-s_0) + R_2(s) - \int_{s_0}^{s} \{pR_2(u)\sin p(s-u) + R_1(u)\cos p(s-u)\}\,du. \tag{5·2·12}$$

Hence $$\Lambda(s)\,\mathrm{H}(s) + \mathrm{H}'(s) = (\Lambda(s)\,K_1 + pK_3)\cos p(s-s_0) + (\Lambda(s)\,K_3 - pK_1)\sin p(s-s_0) + R_2(s) - \frac{1}{p}\int_{s_0}^{s} \{p^2R_2(u) + \Lambda(s)\,R_1(u)\}\sin p(s-u)\,du - \int_{s_0}^{s} \{R_1(u) - \Lambda(s)\,R_2(u)\}\cos p(s-u)\,du, \tag{5·2·13}$$

and

$$\mathrm{e}^{-P(u)}\{p^2R_2(u) + \Lambda(s)\,R_1(u)\} = \left\{w_1'(u) - \frac{g\cos\alpha}{V_u}\right\}\{p^2 + \Lambda(u)\,\Lambda(s)\} - p^2T_2(u) - \Lambda(s)\,T_3(u), \tag{5·2·14}$$

$$\mathrm{e}^{-P(u)}\{R_1(u) - \Lambda(s)\,R_2(u)\} = \left\{w_1'(u) - \frac{g\cos\alpha}{V_u}\right\}\{\Lambda(u) - \Lambda(s)\} - T_3(u) + \Lambda(s)\,T_2(u). \tag{5·2·15}$$

The yaw may be obtained from (5·2·10), since

$$\Xi = \frac{1}{V}\{\mathrm{H}(s)\,\mathrm{e}^{-P(s)} - w_1(s)\}.$$

Also $d\zeta/dt$ is given by (4·4·7) and (5·2·13). The angular deviation Z is then obtained as a result of a single integration from (4·3·9).

In the succeeding sections we use the formulae just obtained to derive separate explicit solutions for Z, Ξ and $d\zeta/dt$ in the ten cases (2) to (11) mentioned at the end of § 4·4.

5·3. Initial yaw

We take $$\Xi = \Xi_0, \quad d\zeta/dt = 0, \quad \mathrm{Z} = 0$$

at launch, and assume no disturbing factors. Then, by (5·2·1, 8, 9), $R(s) = 0$ and

$$K_1 = V_0\Xi_0, \quad K_2 = -\frac{\Lambda_0}{p}V_0\Xi_0.$$

Hence, by (5·2·7), $$\mathrm{H}(s) = V_0\Xi_0\left\{\cos p(s-s_0) - \frac{\Lambda_0}{p}\sin p(s-s_0)\right\}, \tag{5·3·1}$$

and therefore $$\mathrm{H}'(s) = -pV_0\Xi_0\left\{\sin p(s-s_0) + \frac{\Lambda_0}{p}\cos p(s-s_0)\right\}. \tag{5·3·2}$$

It follows from these two equations, and from (4·4·6, 7) and (4·3·9), that

$$\Xi = \Xi_0\frac{V_0}{V}\mathrm{e}^{-P(s)}\left\{\cos p(s-s_0) - \frac{\Lambda_0}{p}\sin p(s-s_0)\right\}, \tag{5·3·3}$$

$$\frac{d\zeta}{dt} = \Xi_0 V_0 \,\mathrm{e}^{-P(s)}\left[\{\Lambda(s)-\Lambda(s_0)\}\cos p(s-s_0) - \frac{1}{p}\{p^2+\Lambda(s)\,\Lambda(s_0)\}\sin p(s-s_0)\right], \quad (5\cdot3\cdot4)$$

and

$$\mathrm{Z} = \Xi_0 V_0\left[\frac{1}{V_0} - \frac{\mathrm{e}^{-P(s)}}{V}\left\{\cos p(s-s_0) - \frac{\Lambda_0}{p}\sin p(s-s_0)\right\} - \frac{1}{p}\int_{s_0}^{s}\{p^2+\Lambda(u)\,\Lambda(s_0)\}\,\mathrm{e}^{-P(u)}\sin p(u-s_0)\frac{du}{V_u}\right.$$
$$\left. + \int_{s_0}^{s}\mathrm{e}^{-P(u)}\{\Lambda(u)-\Lambda(s_0)\}\cos p(u-s_0)\frac{du}{V_u}\right]. \quad (5\cdot3\cdot5)$$

5·4. Initial rate of turn

We take $\Xi = \mathrm{Z} = 0, \quad d\zeta/dt = \zeta_{01}$

at launch, and assume no disturbing factors. Then, by (5·2·1, 8, 9), $R(s) = 0$, and

$$K_1 = 0, \quad K_2 = \frac{1}{p}\zeta_{01}.$$

Hence, by (5·2·7),

$$\mathrm{H}(s) = \frac{1}{p}\zeta_{01}\sin p(s-s_0), \quad (5\cdot4\cdot1)$$

and therefore

$$\mathrm{H}'(s) = \zeta_{01}\cos p(s-s_0). \quad (5\cdot4\cdot2)$$

It follows from these two equations, and from (4·4·6, 7) and (4·3·9), that

$$\Xi = \zeta_{01}\frac{\mathrm{e}^{-P(s)}}{pV}\sin p(s-s_0), \quad (5\cdot4\cdot3)$$

$$\frac{d\zeta}{dt} = \zeta_{01}\,\mathrm{e}^{-P(s)}\left\{\cos p(s-s_0) + \frac{\Lambda(s)}{p}\sin p(s-s_0)\right\}, \quad (5\cdot4\cdot4)$$

and

$$\mathrm{Z} = \zeta_{01}\left[-\frac{\mathrm{e}^{-P(s)}}{pV}\sin p(s-s_0) + \int_{s_0}^{s}\mathrm{e}^{-P(u)}\left\{\cos p(u-s_0) + \frac{\Lambda(u)}{p}\sin p(u-s_0)\right\}\frac{du}{V_u}\right]. \quad (5\cdot4\cdot5)$$

5·5. Gravity

We consider here the case (4) mentioned at the end of § 4·4, i.e. we determine the effect of gravity upon the motion assuming perfect launch and that no other disturbing forces act. The effect of gravity tip-off (see § 3·7) may be determined from (4·4·12), §§ 5·3 and 5·4 when the appropriate initial conditions are known. We have, from (5·2·4, 5, 8, 11),

$$R_1(s) = -\Lambda(s)\,\mathrm{e}^{P(s)}\frac{g\cos\alpha}{V}, \quad R_2(s) = -\mathrm{e}^{P(s)}\frac{g\cos\alpha}{V},$$

$$K_1 = K_3 = 0.$$

Hence, by (5·2·10, 12, 13),

$$\mathrm{H}(s) = -g\cos\alpha\int_{s_0}^{s}\mathrm{e}^{P(u)}\left\{\cos p(s-u) - \frac{\Lambda(u)}{p}\sin p(s-u)\right\}\frac{du}{V_u}, \quad (5\cdot5\cdot1)$$

$$\mathrm{H}'(s) = -g\cos\alpha\left[\frac{\mathrm{e}^{P(s)}}{V} - p\int_{s_0}^{s}\mathrm{e}^{P(u)}\left\{\sin p(s-u) + \frac{\Lambda(u)}{p}\cos p(s-u)\right\}\frac{du}{V_u}\right], \quad (5\cdot5\cdot2)$$

and

$$\Lambda(s)\,\mathrm{H}(s) + \mathrm{H}'(s) = -g\cos\alpha\left[\frac{\mathrm{e}^{P(s)}}{V} - \frac{1}{p}\int_{s_0}^{s}\{p^2+\Lambda(u)\,\Lambda(s)\}\,\mathrm{e}^{P(u)}\sin p(s-u)\frac{du}{V_u}\right.$$
$$\left. - \int_{s_0}^{s}\{\Lambda(u)-\Lambda(s)\}\,\mathrm{e}^{P(u)}\cos p(s-u)\frac{du}{V_u}\right]. \quad (5\cdot5\cdot3)$$

It follows from these three equations, and from (4·4·6, 7) and (4·3·9), that

$$\Xi = -\frac{g\cos\alpha}{V}\,\mathrm{e}^{-P(s)}\int_{s_0}^{s}\mathrm{e}^{P(u)}\left\{\cos p(s-u)-\frac{\Lambda(u)}{p}\sin p(s-u)\right\}\frac{du}{V_u}, \tag{5·5·4}$$

$$\frac{d\zeta}{dt} = g\cos\alpha\,\mathrm{e}^{-P(s)}\left[\frac{1}{p}\int_{s_0}^{s}\{p^2+\Lambda(u)\,\Lambda(s)\}\,\mathrm{e}^{P(u)}\sin p(s-u)\frac{du}{V_u}+\int_{s_0}^{s}\{\Lambda(u)-\Lambda(s)\}\,\mathrm{e}^{P(u)}\cos p(s-u)\frac{du}{V_u}\right], \tag{5·5·5}$$

and

$$\begin{aligned}Z = g\cos\alpha\Bigg[&\frac{\mathrm{e}^{-P(s)}}{V}\int_{s_0}^{s}\mathrm{e}^{P(u)}\left\{\cos p(s-u)-\frac{\Lambda(u)}{p}\sin p(s-u)\right\}\frac{du}{V_u}\\&+\frac{1}{p}\int_{s_0}^{s}\frac{\mathrm{e}^{-P(u)}}{V_u}\,du\int_{s_0}^{u}\{p^2+\Lambda(u)\,\Lambda(v)\}\,\mathrm{e}^{P(v)}\sin p(u-v)\frac{dv}{V_v}\\&+\int_{s_0}^{s}\frac{\mathrm{e}^{-P(u)}}{V_u}\,du\int_{s_0}^{u}\{\Lambda(v)-\Lambda(u)\}\,\mathrm{e}^{P(v)}\cos p(u-v)\frac{dv}{V_v}\Bigg].\end{aligned} \tag{5·5·6}$$

When the rocket is 'infinitely stable' (i.e. $n^2 = p^2 = \infty$) so that the axis is always tangent to the trajectory, equations (5·5·4, 5, 6) reduce to

$$\Xi = 0, \quad \frac{d\zeta}{dt} = \frac{g\cos\alpha}{V}, \quad Z = g\cos\alpha\int_{s_0}^{s}\frac{du}{V_u^2}. \tag{5·5·7}$$

On the other hand, if the rocket is not infinitely stable, the deviation may be quite different from that given in (5·5·7). For example, when the rocket is neutrally stable and unrotated ($n^2 = 0$, $\gamma = 0$), and when κ and all the aerodynamic lateral forces and couples can be neglected, we obtain

$$\Xi = Z = \frac{g\cos\alpha}{V}(t-t_0), \quad d\zeta/dt = 0. \tag{5·5·8}$$

5·6. Wind

It was shown in § 3·52 that the effect of the wind component along the direction of projection **OZ** may be neglected during the burning period of the rocket. Accordingly, we need only consider the cross wind perpendicular to this direction. If w_L and w_F are the components of the wind speed from left to right across the line of fire, and along the line of fire, then

$$w_1 = w\sin\xi\,\mathrm{e}^{i\xi_1} = w_F\sin\alpha - iw_L \tag{5·6·1}$$

when the wind speed has no vertical component.

We consider here the case (5) mentioned at the end of § 4·4, i.e. perfect launch and no other disturbing factors are assumed. Hence, by (5·2·4, 5, 8, 11),

$$R_1(s) = w_1'(s)\,\mathrm{e}^{P(s)}\Lambda(s), \quad R_2(s) = w_1'(s)\,\mathrm{e}^{P(s)}, \tag{5·6·2}$$

$$K_1 = w_1(s_0), \qquad K_3 = -\frac{1}{p}\Lambda(s_0)\,w_1(s_0). \tag{5·6·3}$$

It follows, from (5.2.10, 12, 13), that

$$\begin{aligned}H(s) = w_1(s_0)&\left\{\cos p(s-s_0)-\frac{\Lambda(s_0)}{p}\sin p(s-s_0)\right\}\\&+\int_{s_0}^{s}w_1'(u)\,\mathrm{e}^{P(u)}\left\{\cos p(s-u)-\frac{\Lambda(u)}{p}\sin p(s-u)\right\}du,\end{aligned} \tag{5·6·4}$$

$$\mathrm{H}'(s) = -pw_1(s_0)\left\{\sin p(s-s_0)+\frac{\Lambda(s_0)}{p}\cos p(s-s_0)\right\}+w_1'(s)\,\mathrm{e}^{P(s)}$$
$$-p\int_{s_0}^{s} w_1'(u)\,\mathrm{e}^{P(u)}\left\{\sin p(s-u)+\frac{\Lambda(u)}{p}\cos p(s-u)\right\}du, \tag{5·6·5}$$

and

$$\Lambda(s)\,\mathrm{H}(s)+\mathrm{H}'(s) = w_1(s_0)\left[\{\Lambda(s)-\Lambda(s_0)\}\cos p(s-s_0)-\frac{1}{p}\{p^2+\Lambda(s)\,\Lambda(s_0)\}\sin p(s-s_0)\right]$$
$$+\int_{s_0}^{s} w_1'(u)\,\mathrm{e}^{P(u)}\{\Lambda(s)-\Lambda(u)\}\cos p(s-u)\,du+w_1'(s)\,\mathrm{e}^{P(s)}$$
$$-\frac{1}{p}\int_{s_0}^{s} w_1'(u)\,\mathrm{e}^{P(u)}\{p^2+\Lambda(u)\,\Lambda(s)\}\sin p(s-u)\,du. \tag{5·6·6}$$

Accordingly, by (4·4·6, 7) and (4·3·9),

$$\Xi = \frac{w_1(s_0)}{V}\,\mathrm{e}^{-P(s)}\left\{\cos p(s-s_0)-\frac{\Lambda(s_0)}{p}\sin p(s-s_0)\right\}-\frac{w_1(s)}{V}$$
$$+\frac{\mathrm{e}^{-P(s)}}{V}\int_{s_0}^{s} w_1'(u)\,\mathrm{e}^{P(u)}\left\{\cos p(s-u)-\frac{\Lambda(u)}{p}\sin p(s-u)\right\}du, \tag{5·6·7}$$

$$\frac{d\zeta}{dt} = w_1(s_0)\,\mathrm{e}^{-P(s)}\left[\{\Lambda(s)-\Lambda(s_0)\}\cos p(s-s_0)-\frac{1}{p}\{p^2+\Lambda(s)\,\Lambda(s_0)\}\sin p(s-s_0)\right]$$
$$+\mathrm{e}^{-P(s)}\int_{s_0}^{s} w_1'(u)\,\mathrm{e}^{P(u)}\{\Lambda(s)-\Lambda(u)\}\cos p(s-u)\,du$$
$$-\frac{1}{p}\mathrm{e}^{-P(s)}\int_{s_0}^{s} w_1'(u)\,\mathrm{e}^{P(u)}\{p^2+\Lambda(s)\,\Lambda(u)\}\sin p(s-u)\,du, \tag{5·6·8}$$

and
$$\mathrm{Z} = \frac{w_1(s)}{V}-\frac{w_1(s_0)}{V}\,\mathrm{e}^{-P(s)}\left\{\cos p(s-s_0)-\frac{\Lambda(s_0)}{p}\sin p(s-s_0)\right\}$$
$$+w_1(s_0)\int_{s_0}^{s}\mathrm{e}^{-P(u)}\left[\{\Lambda(u)-\Lambda(s_0)\}\cos p(u-s_0)-\frac{1}{p}\{p^2+\Lambda(u)\,\Lambda(s_0)\}\sin p(u-s_0)\right]du$$
$$-\frac{\mathrm{e}^{-P(s)}}{V}\int_{s_0}^{s} w_1'(u)\,\mathrm{e}^{P(u)}\left\{\cos p(s-u)-\frac{\Lambda(u)}{p}\sin p(s-u)\right\}du$$
$$+\int_{s_0}^{s}\mathrm{e}^{-P(u)}\frac{du}{V_u}\int_{s_0}^{u} w_1'(v)\,\mathrm{e}^{P(v)}\{\Lambda(u)-\Lambda(v)\}\cos p(u-v)\,dv$$
$$-\frac{1}{p}\int_{s_0}^{s}\mathrm{e}^{-P(u)}\frac{du}{V_u}\int_{s_0}^{u} w_1'(v)\,\mathrm{e}^{P(v)}\{p^2+\Lambda(u)\,\Lambda(v)\}\sin p(u-v)\,dv. \tag{5·6·9}$$

In these formulae the derivative of w_1 with respect to the arc length occurs in several places under the sign of integration. It is possible, of course, to avoid this by integrating by parts, but the formulae are in many ways more convenient as they stand, since for a constant wind speed $w_1'(s) = 0$, and if the wind varies logarithmically with the height (see § 6·8) the first derivative is more convenient to manipulate.

5·7. Tolerances

In this section we consider the effect of the six tolerance angles

$$\alpha_C,\quad \alpha_G,\quad \alpha_L,\quad \alpha_M,\quad \alpha_N,\quad \alpha_R$$

upon the motion of the rocket during the burning period when there are no other disturbing factors and when the projectile is perfectly launched. This corresponds to cases (6) to (11) of § 4·4. We consider the combined effect of all six tolerances. The contribution due to any particular tolerance can easily be picked out from the formulae (see, for example, §§ 6·102 to 6·107). The twelve angles α_P, ϕ_P $(P = C, G, L, M, N, R)$ will, in general, vary during burning.

We have, by (5·2·4, 5, 8, 11),

$$R_1(s) = -e^{P(s)}\,T_3(s), \quad R_2(s) = -e^{P(s)}\,T_2(s), \tag{5·7·1}$$

$$K_1 = 0, \qquad K_3 = -\frac{1}{p}\mu_3(s_0)\,e^{i\sigma_0}. \tag{5·7·2}$$

Hence, by (5·2·10, 12, 13),

$$H(s) = -\frac{1}{p}\mu_3(s_0)\,e^{i\sigma_0}\sin p(s-s_0) - \frac{1}{p}\int_{s_0}^{s} e^{P(u)}\{p\,T_2(u)\cos p(s-u) - T_3(u)\sin p(s-u)\}\,du, \tag{5·7·3}$$

$$H'(s) = -\mu_3(s_0)\,e^{i\sigma_0}\cos p(s-s_0) - e^{P(s)}T_2(s) + \int_{s_0}^{s} e^{P(u)}\{p\,T_2(u)\sin p(s-u) + T_3(u)\cos p(s-u)\}\,du, \tag{5·7·4}$$

and

$$\Lambda(s)\,H(s) + H'(s) = -\mu_3(s_0)\,e^{i\sigma_0}\left\{\cos p(s-s_0) + \frac{\Lambda(s)}{p}\sin p(s-s_0)\right\} - e^{P(s)}\,T_2(s)$$
$$+\frac{1}{p}\int_{s_0}^{s} e^{P(u)}\{p^2 T_2(u) + \Lambda(s)\,T_3(u)\}\sin p(s-u)\,du$$
$$+\int_{s_0}^{s} e^{P(u)}\{T_3(u) - \Lambda(s)\,T_2(u)\}\cos p(s-u)\,du. \tag{5·7·5}$$

Let $\mu(s)$ be an arbitrary integrable function of s. We define nine functions of $\mu(s)$, and therefore of s, in the following way:

$$x_1(s,\mu) = x_1(\mu) = -\frac{e^{-P(s)}}{pV}\int_{s_0}^{s}\mu(u)\,e^{P(u)+i\sigma(u)}\{p\cos p(s-u) - \Lambda(u)\sin p(s-u)\}\,dV_u, \tag{5·7·6}$$

$$x_2(s,\mu) = x_2(\mu) = \frac{e^{-P(s)}}{pV}\int_{s_0}^{s}\mu(u)\,e^{P(u)+i\sigma(u)}\sin p(s-u)\,dV_u, \tag{5·7·7}$$

$$x_3(s,\mu) = x_3(\mu) = -\frac{e^{-P(s)}}{pV}\,\mu(s_0)\,e^{i\sigma_0}\sin p(s-s_0)$$
$$+\frac{e^{-P(s)}}{pV}\int_{s_0}^{s}\mu(u)\,e^{P(u)+i\sigma(u)}\{p\cos p(s-u) - P'(u)\sin p(s-u)\}\,du, \tag{5·7·8}$$

$$y_1(\mu) = \frac{e^{-P(s)}}{p}\int_{s_0}^{s}\mu(u)\,\{p^2 + \Lambda(u)\,\Lambda(s)\}\,e^{P(u)+i\sigma(u)}\sin p(s-u)\,dV_u$$
$$+e^{-P(s)}\int_{s_0}^{s}\mu(u)\,\{\Lambda(u) - \Lambda(s)\}\,e^{P(u)+i\sigma(u)}\cos p(s-u)\,dV_u, \tag{5·7·9}$$

$$y_2(\mu) = \frac{e^{-P(s)}}{p}\int_{s_0}^{s}\mu(u)\,\Lambda(s)\,e^{P(u)+i\sigma(u)}\sin p(s-u)\,dV_u + e^{-P(s)}\int_{s_0}^{s}\mu(u)\,e^{P(u)+i\sigma(u)}\cos p(s-u)\,dV_u, \tag{5·7·10}$$

$$y_3(\mu) = -\mu(s_0)\,e^{i\sigma_0-P(s)}\left\{\cos p(s-s_0) + \frac{\Lambda(s)}{p}\sin p(s-s_0)\right\} + \mu(s)\,e^{i\sigma(s)}$$
$$-\frac{e^{-P(s)}}{p}\int_{s_0}^{s}\mu(u)\,e^{P(u)+i\sigma(u)}\{p^2 + \Lambda(s)\,P'(u)\}\sin p(s-u)\,du$$
$$-e^{-P(s)}\int_{s_0}^{s}\mu(u)\,e^{P(u)+i\sigma(u)}\{P'(u) - \Lambda(s)\}\cos p(s-u)\,du. \tag{5·7·11}$$

We define, for $\nu = 1, 2$ and 3,

$$z_\nu(s,\mu) = z_\nu(\mu) = -x_\nu(\mu) + \int_{s_0}^{s} y_\nu(u,\mu)\frac{du}{V_u}. \tag{5·7·12}$$

Hence

$$z_1(\mu) = \frac{e^{-P(s)}}{pV}\int_{s_0}^{s}\mu(u)\, e^{P(u)+i\sigma(u)}\{p\cos p(s-u) - \Lambda(u)\sin p(s-u)\}\, dV_u$$
$$+\frac{1}{p}\int_{s_0}^{s} e^{-P(u)}\frac{du}{V_u}\int_{s_0}^{u}\mu(v)\,\{p^2+\Lambda(u)\,\Lambda(v)\}\, e^{P(v)+i\sigma(v)}\sin p(u-v)\, dV_v$$
$$+\int_{s_0}^{s} e^{-P(u)}\frac{du}{V_u}\int_{s_0}^{u}\mu(v)\,\{\Lambda(v)-\Lambda(u)\}\, e^{P(v)+i\sigma(v)}\cos p(u-v)\, dV_v, \tag{5·7·13}$$

$$z_2(\mu) = -\frac{e^{-P(s)}}{pV}\int_{s_0}^{s}\mu(u)\, e^{P(u)+i\sigma(u)}\sin p(s-u)\, dV_u$$
$$+\frac{1}{p}\int_{s_0}^{s} e^{-P(u)}\frac{du}{V_u}\int_{s_0}^{u}\mu(v)\,\Lambda(u)\, e^{P(v)+i\sigma(v)}\sin p(u-v)\, dV_v$$
$$+\int_{s_0}^{s} e^{-P(u)}\frac{du}{V_u}\int_{s_0}^{u}\mu(v)\, e^{P(v)+i\sigma(v)}\cos p(u-v)\, dV_v, \tag{5·7·14}$$

and

$$z_3(\mu) = \frac{e^{-P(s)}}{pV}\mu(s_0)\, e^{i\sigma_0}\sin p(s-s_0) + \int_{s_0}^{s}\mu(u)\, e^{i\sigma(u)}\frac{du}{V_u}$$
$$-\frac{e^{-P(s)}}{pV}\int_{s_0}^{s}\mu(u)e^{P(u)+i\sigma(u)}\{p\cos p(s-u) - P'(u)\sin p(s-u)\}\, du$$
$$-\mu(s_0)\, e^{i\sigma_0}\int_{s_0}^{s} e^{-P(u)}\left\{\cos p(u-s_0) + \frac{\Lambda(u)}{p}\sin p(u-s_0)\right\}\frac{du}{V_u}$$
$$-\frac{1}{p}\int_{s_0}^{s} e^{-P(u)}\frac{du}{V_u}\int_{s_0}^{u}\mu(v)\, e^{P(v)+i\sigma(v)}\{p^2+\Lambda(u)\,P'(v)\}\sin p(u-v)\, dv$$
$$-\int_{s_0}^{s} e^{-P(u)}\frac{du}{V_u}\int_{s_0}^{u}\mu(v)\, e^{P(v)+i\sigma(v)}\{P'(v)-\Lambda(u)\}\cos p(u-v)\, dv. \tag{5·7·15}$$

It follows from these definitions, and from (5·7·3, 4, 5), that

$$\Xi = \sum_{\nu=1}^{3} x_\nu(\mu_\nu), \tag{5·7·16}$$

$$\frac{d\zeta}{dt} = \sum_{\nu=1}^{3} y_\nu(\mu_\nu), \tag{5·7·17}$$

and

$$Z = \sum_{\nu=1}^{3} z_\nu(\mu_\nu). \tag{5·7·18}$$

These formulae and the definitions of μ_1, μ_2 and μ_3 can be used to obtain expressions for Ξ, $d\zeta/dt$ and Z for each of the six tolerances separately, perfect launch being assumed.

6. SIMPLIFIED SOLUTIONS FOR A SPIN PROPORTIONAL TO THE VELOCITY

6·1. ASSUMPTIONS

In this section the solutions obtained in § 5 are simplified and brought into forms suitable for quick numerical calculation. In order to do this it is necessary to make a number of additional assumptions, namely, Assumptions C 1, 2 of § 3·63. Assumption C 3 is only used in §§ 6·9, 6·10.

The Assumptions B and C may be grouped into two main categories:

(*a*) *Quantities assumed to be constant*

(i) Parameters connected with the change of mass and rate of consumption of the propellant, and with the aerodynamic forces, namely f, β, n^2.

(ii) Other parameters connected with various asymmetries in designs and functioning (see Assumption C 3).

(*b*) *Quantities assumed to be negligible*

(i) Parameters contributing to the damping, namely κ, α_1, α_2.

(ii) Other parameters.

Of these, (*a*) (i) and (*b*) (i) are the most important. For (*a*) (ii) see § 3·63 and §§ 6·9, 6·10; (*b*) (ii) requires no comment except as regards those parameters which depend upon the Magnus forces. It is believed that the other parameters included are small in all practical cases and that they can be neglected in comparison with other larger quantities.

We now consider the legitimacy of the assumptions regarding the groups (*a*) (i), (*b*) (i) and the Magnus forces.

The acceleration. The form of the acceleration depends upon the rate of burning of the charge, since this determines both the magnitude of the thrust and the mass of the rocket at any instant. When the charge design is such that the rate of burning is approximately uniform between launch and burnt, the assumption of constant f ($= QW/m$) is justified,† provided, of course, that the ratio m_1/m_{00} is not too small. In many cases, however, and particularly when the temperature of the charge is high, the acceleration will decrease more or less irregularly between launch and burnt.

The aerodynamic coefficient n^2. This parameter determines the wave-length of the oscillations in yaw. The chief factors causing variations in n^2 during burning are (i) the decrease in mass due to the burning of the charge, and (ii) the velocity of the air relative to the projectile, and of these the latter is the more important.

The decrease in mass affects the position of the centre of gravity and therefore d_1. It also causes the transverse moment of inertia A to decrease. Usually, when the charge is situated in the rear portion of the projectile, the centre of gravity will move forward in the rocket during burning, so that d_1 will increase. The charges in n^2 due to (i) are, however, unlikely to be appreciable unless the stability is critical ($n^2 \doteqdot 0$) and the charge-mass ratio is very high.

The main cause of variation in n^2 is the varying velocity of the projectile during burning. This affects the Mach number so that the magnitude of the moment coefficient kd_1 may change appreciably as the velocity passes through that of sound (see § 3·518). For subsonic values kd_1 may be expected to remain approximately constant (apart from the effect of (i)). As the velocity passes through that of sound kd_1 usually decreases‡ sharply to a minimum after which it may increase slowly for increasing Mach number greater than unity.

The ratio β. This is affected only by the change in mass and location of the centre of gravity. For some designs it may increase during burning, and for others it may decrease, but the amount by which it changes is usually very small unless the charge-mass ratio is very high.

† In this case the form of the thrust-time curve is approximately trapezoidal, the sides of the trapezium being steep.

‡ I.e. the *destabilizing* moment *increases*.

From the above remarks on f, n^2 and β it is clear that the magnitudes of the variations possible during burning cannot be stated definitely in any detail since they depend to a considerable extent upon the design and performance of the particular rocket under consideration. For some designs the variations may be small, while for others they may be appreciable. In the latter case the justification for the assumption of no variation rests on the fact that it is in the initial stages of flight, while the velocity is subsonic, that the variation is least, and it is during this period that the greater part of the deviation of the rocket is, in general, built up, the motion during the later stages of burning consisting mainly of oscillations of decreasing amplitude about a fixed angular deviation. Accordingly, the values of f, n^2 and β at launch may be taken as the constant values. This argument does not, of course, apply when the launching velocity is in the region of the velocity of sound, or when the variations mentioned are large. In such cases the solutions obtained under the assumptions cannot be expected to be valid approximations, particularly if the stability is critical, and it may be necessary to split up the trajectory into a number of arcs in each of which the assumptions hold.

Neglect of damping terms. Damping of the oscillations is due to two causes: (i) the effect of the burning gases ejected from the projectile, and (ii) the effect of certain of the aerodynamic forces and couples. The increase in velocity causes most of the decrease in the amplitude of the yaw during burning, but it is convenient not to consider this as a damping since the relevant quantity in the equations is $V\Xi$ rather than Ξ. The cause (i) contributes the parameter κ and (ii) contributes α_1 and α_2. The three quantities are usually of roughly the same order of magnitude. Their neglect amounts to putting

$$\Re P(s) = \Re \Lambda(s) = 0,$$

where $\Re$ denotes real part.

The effect of damping is greatest upon the yaw, and it is not really justifiable to neglect it except when the time of burning is short. For long times of burning where several oscillations in yaw occur there may be an appreciable decrease in amplitude—apart from that due to the increasing velocity—between the initial and final oscillations, which is attributable to damping.

From a small number of calculations carried out with and without damping, it is thought that the effect of damping upon the angular deviation is not very appreciable, but this conclusion cannot be stated firmly without further confirmation. The main justification for the neglect of the effect of damping upon the angular deviation is the fact, mentioned before, that it is in the early stages of flight while the magnitude of the damping terms is not appreciable that the greater part of the angular deviation is built up.

Neglect of Magnus forces. The neglect of the quantities $\beta-\beta_1$, $\beta-\beta_2$ and ϖ means that the effect of the Magnus force $\mathbf{L}_3$ and Magnus couples $\mathbf{M}_3$ and $\mathbf{M}_4$ can be ignored.† Of these the couple $\mathbf{M}_3$ is probably the most appreciable. Measurements of the associated parameter‡ ϖ have been made for certain shells by means of jump-card trials, but no direct measurements have been made for any rocket so far as the author is aware. However, it has been possible to deduce the approximate order of magnitude of ϖ for certain rockets from their behaviour

† The Magnus force $\mathbf{L}_4$ has already been neglected as a result of Assumption A 13.

‡ The parameter ϖ is connected with Fowler's coefficient γ in the following way:

$$\gamma = -V\varpi/2\beta.$$

with different sizes of fins and at different spins. From the meagre data available from these and other sources it is concluded that the effect of the Magnus forces and couples is small except when the projectile is very long in relation to its calibre, and when the external surface is such as to increase the aerodynamic circulation about the axis at the extremities, e.g. if the projectile has fins at the rear end and is spinning very rapidly. In both these cases the moment arm d_3 is large and therefore ϖ may not be negligible.

6·2. General

We assume that the spin is proportional to the velocity, as in § 5. Then

$$r = \gamma V, \tag{6·2·1}$$

where γ is a constant. This includes the case of no spin. It follows from (5·1·3, 4) and Assumptions C 1 and 2 that

$$G(s) = p^2 = n^2 + \beta^2\gamma^2 = -\nu^2 + \beta^2\gamma^2. \tag{6·2·2}$$

Clearly $G(s)$ and p^2 are negative if $\beta^2\gamma^2 < \nu^2$ and, accordingly, the solutions of equation (4·3·8) will be unstable in this case, i.e. the yaw will build up like $e^{|p|s}$ along the trajectory so that Assumption A 1 may no longer hold. For this reason we restrict ourselves to values of γ for which

$$\beta^2\gamma^2 > \nu^2. \tag{6·2·3}$$

This is, obviously, only a restriction if the aerodynamic lift moment $\mathbf{M}_1$ is destabilizing, i.e. if $n^2 < 0$. Further, we shall only consider values of γ corresponding to clockwise rotation as viewed from the rear, i.e. we take

$$\gamma \geqslant 0. \tag{6·2·4}$$

This does not involve any loss of generality, since a reversal of the sign of γ corresponds merely to a reversal of that component of deviation, yaw, etc., which is perpendicular to the plane in which the disturbing factor under consideration (see end of § 4·4) acts at launch.

By virtue of (6·2·3, 4) it follows that the allowable ranges of γ are

$$\left.\begin{aligned} \gamma \geqslant 0 \quad &\text{for} \quad n^2 > 0, \\ \gamma > \frac{\nu}{\beta} \quad &\text{for} \quad n^2 \leqslant 0. \end{aligned}\right\} \tag{6·2·5}$$

In the second case the stability factor is defined to be

$$\left(\frac{\beta\gamma}{\nu}\right)^2 > 1.$$

The two ranges can be expressed conveniently in terms of p which we may take to denote the *positive* root of p^2, namely,

$$\left.\begin{aligned} &\text{stabilizing lift moment } \mathbf{M}_1: \quad p > \beta\gamma, \\ &\text{destabilizing lift moment } \mathbf{M}_1: \quad 0 < p \leqslant \beta\gamma. \end{aligned}\right\} \tag{6·2·6}$$

It will be noted that we include the case of neutral stability ($n^2 = 0$) under a destabilizing moment.

The formulae we obtain are of two kinds, accurate and approximate formulae. The former may be applied for every γ within the ranges (6·2·5), but the latter are only valid for values of γ which are not too small.

In all cases separate formulae are given for the case of no spin $\gamma = 0$. In this case the motion in each of the eleven cases mentioned in § 4·4 is two-dimensional by virtue of the Assumptions A, B and C, and we may regard the plane ZOX in figure 3 as the plane in which the disturbing factor acts at launch, instead of the vertical plane, and write

$$Z = \Theta, \quad \zeta = \theta, \quad \Xi = \delta = \theta - \Theta.$$

The relation between the various directions is illustrated in figure 4 for this case. Also, by (6·2·5), we must have $n^2 > 0$ in this case, although the formulae can easily be extended to include the case $n = 0$.

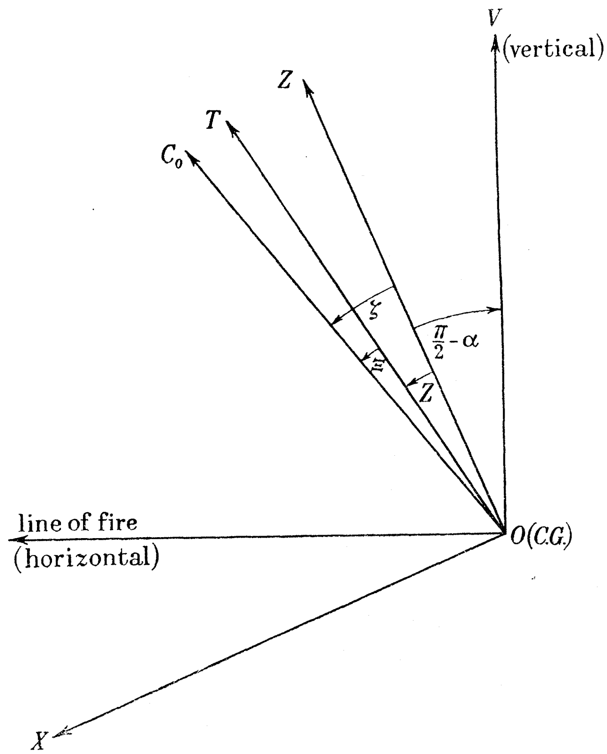

FIGURE 4. Two-dimensional motion.

OZ = Direction of projection.
OT = Tangent to trajectory.
OC_0 = Axis.

Angle $Z = \Theta$ (Angular deviation).
Angle $\zeta = \theta$ (Deviation of Axis).
Angle $\Xi = \gamma$ (Yaw).

6·3. NOTATION

We have, by (5·1·9, 10) and the assumptions,

$$P(s) = -i\beta\gamma(s - s_0), \tag{6·3·1}$$

and

$$\Lambda(s) = i\beta\gamma. \tag{6·3·2}$$

Write

$$\lambda_1 = \beta + \frac{p}{\gamma}, \quad \lambda_2 = \beta - \frac{p}{\gamma}, \tag{6·3·3}$$

$$G_0 = \sqrt{\left(\frac{\gamma}{\pi f}\right)} V_0, \quad G_1 = \sqrt{\left(\frac{\gamma}{\pi f}\right)} V, \tag{6·3·4}$$

$$w_{jk} = G_j \sqrt{|\lambda_k|}, \tag{6·3·5}$$

$$G_{jk} = G_j \sqrt{|1 - \lambda_k|}. \tag{6·3·6}$$

Here V_0 and V are the velocities at launch, and at any instant between launch and burnt, respectively.† The suffixes j and k here and later take the values

$$j = 0, 1; \quad k = 1, 2.$$

We also write‡

$$\phi_j = \tfrac{1}{2}\pi G_j^2, \quad \phi = \phi_1 - \phi_0, \tag{6·3·7}$$

$$\psi_{jk} = \tfrac{1}{2}\pi w_{jk}^2, \quad \psi_k = \psi_{1k} - \psi_{0k}. \tag{6·3·8}$$

The quantity λ_2 is positive or negative according as the lift moment is destabilizing or stabilizing (see (6·2·6)). The quantity λ_1 is always positive.

In the case of unrotated motion when $\gamma = 0$, the quantities λ_1 and λ_2 are not required, the values $\lambda_1\gamma$ and $\lambda_2\gamma$ being replaced by n and $-n$ respectively. In this case we have $G_0 = G_1 = 0$, and we write

$$v_0 = \sqrt{\left(\frac{n}{\pi f}\right)} V_0, \quad v_1 = \sqrt{\left(\frac{n}{\pi f}\right)} V, \tag{6·3·9}$$

so that

$$w_{01} = w_{02} = v_0, \quad w_{11} = w_{12} = v_1, \tag{6·3·10}$$

and

$$\phi = 0, \quad \psi_1 = \psi_2 = \tfrac{1}{2}\pi(v_1^2 - v_0^2). \tag{6·3·11}$$

The quantities G_j, G_{jk} arise only in § 6·10.

Owing to Assumption C 1, we have§

$$s - s_0 = \frac{1}{2f}(V^2 - V_0^2), \tag{6·3·12}$$

so that

$$\gamma(s - s_0) = \tfrac{1}{2}\pi(G_1^2 - G_0^2) = \phi,$$

$$(\beta\gamma + p)(s - s_0) = \tfrac{1}{2}\pi(w_{11}^2 - w_{01}^2) = \psi_1,$$

$$(\beta\gamma - p)(s - s_0) = \pm\tfrac{1}{2}\pi(w_{12}^2 - w_{02}^2) = \pm\psi_2,$$

the positive sign being taken when $0 < p \leqslant \beta\gamma$ and the negative when $p > \beta\gamma$. Also

$$n(s - s_0) = \tfrac{1}{2}\pi(v_1^2 - v_0^2)$$

and we have

$$p^2 + \Lambda(u)\Lambda(v) = p^2 - \beta^2\gamma^2 = n^2,$$

$$\Lambda(u) - \Lambda(v) = 0,$$

$$p^2 + \Lambda(u) P'(v) = p^2 + \beta^2\gamma^2,$$

$$P'(v) - \Lambda(u) = -2i\beta\gamma.$$

6·4. Fresnel functions

Write, for any non-negative real u and v,

$$\mathscr{E}(u, v) = C(u, v) + iS(u, v) = \int_u^v e^{\frac{1}{2}\pi i x^2}\, dx, \tag{6·4·1}$$

and

$$\mathscr{E}(u) = \mathscr{E}(0, u) = C(u) + iS(u). \tag{6·4·2}$$

† The formulae obtained will usually be applied at burnt. For this reason, and for convenience in the notation, the suffix 1 is affixed to G in (6·3·4). It should be noted, however, that the formulae hold at all instants during burning for which the assumptions regarding the magnitude of γ are valid.

‡ The quantity ϕ has no connexion with the ϕ defined in § 3·214.

§ It should be noted that it is not assumed that f is constant before launch.

Here $$C(u,v) = C(v)-C(u), \quad S(u,v) = S(v)-S(u), \tag{6·4·3}$$
where $C(u)$ and $S(u)$ are the ordinary Fresnel integrals.†

Owing to the highly oscillatory nature of the functions $C(u)$ and $S(u)$, it is more convenient in numerical work to employ the steadily decreasing functions $A(u)$ and $B(u)$ which are defined as follows:

$$D(u) = B(u)+iA(u) = \mathrm{e}^{-\frac{1}{2}\pi iu^2}\left\{\frac{1+i}{2}-\mathscr{E}(u)\right\} = \mathrm{e}^{-\frac{1}{2}\pi iu^2}\mathscr{E}(u,\infty). \tag{6·4·4}$$

It can be shown, by the methods of contour integration, that

$$A(u) = \frac{1}{\pi\sqrt{2}}\int_0^\infty \mathrm{e}^{-\frac{1}{2}\pi u^2 x}\frac{x^{-\frac{1}{2}}\,dx}{1+x^2}, \quad B(u) = \frac{1}{\pi\sqrt{2}}\int_0^\infty \mathrm{e}^{-\frac{1}{2}\pi u^2 x}\frac{x^{\frac{1}{2}}\,dx}{1+x^2}. \tag{6·4·5}$$

These expressions display the monotonic properties of the two functions. Tables of $A(u)$, $B(u)$ and of the functions $A_1(u)$, $Z(u)$ and $Z_1(u)$ are given in Appendix A. The other functions are defined as follows:

$$A_1(u) = \frac{1}{\pi u}-A(u), \tag{6·4·6}$$

and $$Z(u) = Z_1(u)+\log u = \pi\int_0^u A(x)\,dx. \tag{6·4·7}$$

Here $\log u$ denotes the natural logarithm. The following formulae are occasionally of use:

$$A'(u) = -\pi uB(u), \quad B'(u) = \pi uA(u)-1, \tag{6·4·8}$$

and $$\int_0^u B(x)\,dx = \tfrac{1}{4}-\tfrac{1}{2}\{A^2(u)+B^2(u)\}. \tag{6·4·9}$$

In the remainder of this section the following five functions occur frequently:

$$E(u,v) = \cos\tfrac{1}{2}\pi u^2\,S(u,v)-\sin\tfrac{1}{2}\pi u^2\,C(u,v)-\frac{1}{\pi v}\{1-\cos\tfrac{1}{2}\pi(v^2-u^2)\}, \tag{6·4·10}$$

$$E^*(u,v) = E(u,v)+\frac{1}{\pi v}, \tag{6·4·11}$$

$$G(u,v) = \cos\tfrac{1}{2}\pi u^2\,C(u,v)+\sin\tfrac{1}{2}\pi u^2\,S(u,v)-\frac{1}{\pi v}\sin\tfrac{1}{2}\pi(v^2-u^2), \tag{6·4·12}$$

$$F(u,v) = \tfrac{1}{2}\pi\{C^2(u,v)+S^2(u,v)\}+\frac{1}{v}\{\cos\tfrac{1}{2}\pi v^2\,S(u,v)-\sin\tfrac{1}{2}\pi v^2\,C(u,v)\}, \tag{6·4·13}$$

and $$H(u,v) = \pi\int_u^v du_1\int_u^{u_1}\sin\tfrac{1}{2}\pi(u_1^2-u_2^2)\,du_2+\frac{1}{v}\{\cos\tfrac{1}{2}\pi v^2\,C(u,v)+\sin\tfrac{1}{2}\pi v^2\,S(u,v)\}. \tag{6·4·14}$$

In terms of $A(u)$, $B(u)$, etc., these may be written as

$$E(u,v) = A(u)+A_1(u)\cos\tfrac{1}{2}\pi(v^2-u^2)-B(v)\sin\tfrac{1}{2}\pi(v^2-u^2)-\frac{1}{\pi v}, \tag{6·4·15}$$

$$E^*(u,v) = A(u)+A_1(u)\cos\tfrac{1}{2}\pi(v^2-u^2)-B(v)\sin\tfrac{1}{2}\pi(v^2-u^2), \tag{6·4·16}$$

$$G(u,v) = B(u)-A_1(u)\sin\tfrac{1}{2}\pi(v^2-u^2)-B(v)\cos\tfrac{1}{2}\pi(v^2-u^2), \tag{6·4·17}$$

$$F(u,v) = \tfrac{1}{2}\pi\{A^2(u)+B^2(u)+A^2(v)+B^2(v)\}-\frac{1}{v}A(v)$$
$$-\pi b(u,v)\cos\tfrac{1}{2}\pi(v^2-u^2)-\pi a(u,v)\sin\tfrac{1}{2}\pi(v^2-u^2), \tag{6·4·18}$$

$$H(u,v) = Z(v)-Z(u)-\frac{1}{v}B(v)+\pi a(u,v)\cos\tfrac{1}{2}\pi(v^2-u^2)-\pi b(u,v)\cos\tfrac{1}{2}\pi(v^2-u^2), \tag{6·4·19}$$

† There are two definitions of the integrals C and S in use. Only the functions defined by (6·4·1, 2) are, however, used here.

where
$$a(u,v) = A(u)\,B(v) + B(u)\,A_1(v) \tag{6·4·20}$$
and
$$b(u,v) = B(u)\,B(v) - A(u)\,A_1(v). \tag{6·4·21}$$
Write
$$\begin{aligned} E_k &= \mathrm{e}^{-i\psi_{0k}}\,\mathscr{E}(w_{0k}, w_{1k}) + \frac{i\,\mathrm{e}^{i\psi_k}}{\pi w_{1k}} \\ &= D(w_{0k}) - \mathrm{e}^{i\psi_k}\left\{D(w_{1k}) - \frac{i}{\pi w_{1k}}\right\} \\ &= G(w_{0k}, w_{1k}) + iE^*(w_{0k}, w_{1k}), \end{aligned} \tag{6·4·22}$$
and put
$$P(u,v) = \frac{1}{v}\{A(u)\sin\tfrac{1}{2}\pi(v^2-u^2) + B(u)\cos\tfrac{1}{2}\pi(v^2-u^2) - B(v)\}, \tag{6·4·23}$$
$$Q(u,v) = \frac{1}{v}\{-A(u)\cos\tfrac{1}{2}\pi(v^2-u^2) + B(u)\sin\tfrac{1}{2}\pi(v^2-u^2) + A(v)\}, \tag{6·4·24}$$
so that
$$P(u,v) + iQ(u,v) = \frac{1}{v}\{\overline{D}(u)\,\mathrm{e}^{\frac{1}{2}\pi i(v^2-u^2)} - \overline{D}(v)\},$$
where the bars denote the conjugate complex values.

In the case of three-dimensional motion the expressions which we derive will be of the form
$$Z = Z^{(1)} - Z^{(2)}, \tag{6·4·25}$$
$$\Xi = \Xi^{(1)} - \Xi^{(2)}, \tag{6·4·26}$$
$$\frac{d\zeta}{dt} = \frac{d\zeta^{(1)}}{dt} - \frac{d\zeta^{(2)}}{dt}, \tag{6·4·27}$$
where the two different parts of each expression correspond to the two different modes of precession of wave-lengths
$$\frac{2\pi}{\beta\gamma + p} \quad \text{and} \quad \frac{2\pi}{\beta\gamma - p}.$$
Thus for any given k, $Z^{(k)}$, $\Xi^{(k)}$ and $d\zeta^{(k)}/dt$ are functions of λ_k, w_{0k}, w_{1k}, G_0, G_1, G_{0k} and G_{1k} alone.

Since λ_2 changes sign according as the lift moment is destabilizing or stabilizing, the expressions for $Z^{(2)}$, $\Xi^{(2)}$ and $d\zeta^{(2)}/dt$ will not be the same in the two cases. We shall, however, only give explicit expressions for the case of a destabilizing moment M_1, when $0 < p \leqslant \beta\gamma$. When the moment is stabilizing the changes necessary in $Z^{(2)}$, $\Xi^{(2)}$ and $d\zeta^{(2)}/dt$ may be found from the following table. A bar denotes a conjugate value.

destabilizing lift moment ($0 < p \leqslant \beta\gamma$)	stabilizing lift moment ($p > \beta\gamma$)
w_{j2}^2	$-w_{j2}^2$
ψ_{j2}	$-\psi_{j2}$
ψ_2	$-\psi_2$
$w_{j2}G(w_{02}, w_{12})$	$-w_{j2}G(w_{02}, w_{12})$
$w_{j2}E(w_{02}, w_{12})$	$w_{j2}E(w_{02}, w_{12})$
$w_{j2}E^*(w_{02}, w_{12})$	$w_{j2}E^*(w_{02}, w_{12})$
$w_{j2}E_2$	$-w_{j2}\overline{E}_2$
$F(w_{02}, w_{12})$	$-F(w_{02}, w_{12})$
$H(w_{02}, w_{12})$	$H(w_{02}, w_{12})$
$w_{j2}D(w_{02})$	$-w_{j2}\overline{D}(w_{02})$
$w_{j2}D(w_{12})$	$-w_{j2}\overline{D}(w_{12})$
$w_{j2}A(w_{02})$	$w_{j2}A(w_{02})$
$w_{j2}A(w_{12})$	$w_{j2}A(w_{12})$
$w_{j2}B(w_{02})$	$-w_{j2}B(w_{02})$
$w_{j2}B(w_{12})$	$-w_{j2}B(w_{12})$

All other parameters such as $\beta\gamma+p$, $\beta\gamma-p$, G_{j2} and G_j remain unaltered in the two cases. As an example of the type of change which occurs, suppose that

$$Z^{(2)} = \frac{\Xi_0}{2p}(\beta\gamma+p)\{1+i\pi w_{02}E_2\},$$

when $0<p\leqslant\beta\gamma$. Then for $p>\beta\gamma$ we have

$$Z^{(2)} = \frac{\Xi_0}{2p}(\beta\gamma+p)\{1-i\pi w_{02}\overline{E}_2\}.$$

As stated at the end of § 6·3, the parameters G_j, G_{jk} only occur in the formulae given in § 6·10. By definition,

$$G_{jk} = G_j\surd(|1-\lambda_k|).$$

When $k=2$

$$\gamma(1-\lambda_k) = \gamma(1-\beta)+p$$

which is always positive,† so that

$$G_{j2} = G_j\surd(1-\lambda_2) \tag{6·4·28}$$

in all cases. When $k=1$, however, we have

$$\gamma(1-\lambda_1) = \gamma(1-\beta)-p,$$

and this is negative when, by (6·2·2),

$$\gamma^2(1-2\beta)<n^2.$$

This is impossible when the lift moment is destabilizing since $\beta<\frac{1}{2}$. If, however, the lift moment is stabilizing, so that $n^2>0$, we shall have

$$\left.\begin{aligned} G_{j1} &= G_j\surd(1-\lambda_1) \quad \text{for} \quad \gamma\surd(1-2\beta)\geqslant n,\\ G_{j1} &= G_j\surd(\lambda_1-1) \quad \text{for} \quad \gamma\surd(1-2\beta)<n.\end{aligned}\right\} \tag{6·4·29}$$

The formulae we obtain in § 6·10 will usually only be valid for γ's which are not small, i.e. we shall require $\gamma\gg n$ so that the second case in (6·4·29) does not arise and we may put

$$G_{j1} = G_j\surd(1-\lambda_1) \tag{6·4·30}$$

in all cases.

Neutral stability. When the stability is neutral

$$n^2 = -\nu^2 = 0,$$

so that

$$p = \beta\gamma.$$

In this case

$$\left.\begin{aligned} &w_{j2} = \psi_{j2} = \psi_2 = 0,\\ &G_{j2} = G_j,\\ &G(w_{02}, w_{12}) = 0, \quad E(w_{02}, w_{12}) = 0,\\ &F(w_{02}, w_{12}) = 0, \quad H(w_{02}, w_{12}) = 1-\frac{V_0}{V},\\ &\pi w_{12}E^*(w_{02}, w_{12}) = 1,\\ &\pi w_{12}E_2 = i, \quad \pi w_{02}E_2 = i\frac{V_0}{V}.\end{aligned}\right\} \tag{6·4·31}$$

These relations simplify the formulae obtained in the case $p\leqslant\beta\gamma$.

† Since OC is the longitudinal principal axis of inertia it follows that $C<A$, i.e. $\beta<\frac{1}{2}$. Actually, of course, β will usually be very small in comparison with $\frac{1}{2}$.

6·5. Initial yaw

We take
$$\Xi = \Xi_0, \quad d\zeta/dt = 0, \quad Z = 0$$
at launch, and assume no disturbing factors. Then, by (5·3·3 to 5·3·5) and the formulae of §§ 6·3 and 6·4,
$$Z = -\frac{\Xi_0}{2p}\{(\beta\gamma - p)(1 + i\pi w_{01} E_1) - (\beta\gamma + p)(1 + i\pi w_{02} E_2)\}, \tag{6·5·1}$$
$$\Xi = -\frac{\Xi_0}{2p}\frac{V_0}{V}\{(\beta\gamma - p)\,\mathrm{e}^{i\psi_1} - (\beta\gamma + p)\,\mathrm{e}^{i\psi_2}\}, \tag{6·5·2}$$
and
$$\frac{d\zeta}{dt} = -i\Xi_0\frac{\nu^2 V_0}{2p}\{\mathrm{e}^{i\psi_1} - \mathrm{e}^{i\psi_2}\}, \tag{6·5·3}$$
when $0 < p \leqslant \beta\gamma$. The corresponding formulae for a stabilizing lift moment ($p > \beta\gamma$) may be written down from (6·5·1 to 6·5·3) with the help of the table in § 6·4.

For unrotated motion ($\gamma = 0$) the above formulae become (see end of § 6·2 and figure 4)
$$Z = \Xi_0\{1 - \pi v_0 E^*(v_0, v_1)\}, \tag{6·5·4}$$
$$\Xi = \Xi_0\frac{V_0}{V}\cos\tfrac{1}{2}\pi(v_1^2 - v_0^2), \tag{6·5·5}$$
and
$$\frac{d\zeta}{dt} = -\Xi_0\frac{V_0}{n}\sin\tfrac{1}{2}\pi(v_1^2 - v_0^2). \tag{6·5·6}$$

6·6. Initial rate of turn

We take
$$\Xi = Z = 0, \quad d\zeta/dt = \zeta_{01}$$
at launch, and assume no disturbing factors. Then, by (5·4·3 to 5·4·5) and the formulae of §§ 6·3 and 6·4,
$$Z = \frac{\zeta_{01}}{2pV}\pi(w_{11}E_1 - w_{12}E_2), \tag{6·6·1}$$
$$\Xi = -i\frac{\zeta_{01}}{2pV}(\mathrm{e}^{i\psi_1} - \mathrm{e}^{i\psi_2}), \tag{6·6·2}$$
and
$$\frac{d\zeta}{dt} = \frac{\zeta_{01}}{2p}\{(\beta\gamma + p)\,\mathrm{e}^{i\psi_1} - (\beta\gamma - p)\,\mathrm{e}^{i\psi_2}\}, \tag{6·6·3}$$
when $0 < p \leqslant \beta\gamma$. The corresponding formulae for a stabilizing lift moment ($p > \beta\gamma$) may be written down from (6·6·1 to 6·6·3) with the help of the table in § 6·4.

For unrotated motion ($\gamma = 0$) the above formulae become (see end of § 6·2 and figure 4)
$$Z = \zeta_{01}\sqrt{\left(\frac{\pi}{nf}\right)}\,G(v_0, v_1), \tag{6·6·4}$$
$$\Xi = \zeta_{01}\frac{1}{nV}\sin\tfrac{1}{2}\pi(v_1^2 - v_0^2), \tag{6·6·5}$$
and
$$d\zeta/dt = \zeta_{01}\cos\tfrac{1}{2}\pi(v_1^2 - v_0^2). \tag{6·6·6}$$

6·7. Gravity

The deviation due to gravity alone may be written in the form

$$Z = Z_g + Z_T, \tag{6·7·1}$$

where Z_T is the deviation due to gravity tip-off, i.e. due to those parts of the initial deviation, yaw and rate of turn which are attributable to the action of gravity, and Z_g is the deviation due to gravity acting after launch when perfect launch is assumed. The yaw Ξ and rate of turn $d\zeta/dt$ may each be split into two parts in the same way. The values of Z_T, Ξ_T and $d\zeta_T/dt$ may be found from (4·4·12) and §§ 6·5, 6·6, when the initial conditions are known. In this section we determine Z_g, Ξ_g and $d\zeta_g/dt$ which result from perfect launch. We have, by (5·5·4 to 5·5·6) and the formulae of §§ 6·3 and 6·4,

$$Z_g = \frac{g\cos\alpha}{2pf}[(\beta\gamma+p)\{H(w_{01},w_{12})-iF(w_{02},w_{12})\}-(\beta\gamma-p)\{H(w_{01},w_{11})-iF(w_{01},w_{11})\}], \tag{6·7·2}$$

$$\Xi_g = -\frac{g\cos\alpha}{2pf}\left[\frac{\beta\gamma+p}{w_{12}}\{\overline{D}(w_{02})\,e^{i\psi_2}-\overline{D}(w_{12})\}-\frac{\beta\gamma-p}{w_{11}}\{\overline{D}(w_{01})\,e^{i\psi_1}-\overline{D}(w_{11})\}\right], \tag{6·7·3}$$

and

$$\frac{d\zeta_g}{dt} = -i\frac{\pi g\cos\alpha}{2pV}[(\beta\gamma+p)\,w_{12}\{\overline{D}(w_{02})\,e^{i\psi_2}-\overline{D}(w_{12})\}-(\beta\gamma-p)\,w_{11}\{\overline{D}(w_{01})\,e^{i\psi_1}-\overline{D}(w_{11})\}], \tag{6·7·4}$$

when $0<p\leqslant\beta\gamma$. The corresponding formulae for a stabilizing lift moment ($p>\beta\gamma$) may be written down from (6·7·2 to 6·7·4) with the help of the table in § 6·4.

For unrotated motion ($\gamma = 0$) these formulae become

$$\Xi_g = \Theta_g = \frac{g\cos\alpha}{f}H(v_0, v_1), \tag{6·7·5}$$

$$\Xi_g = \delta_g = -\frac{g\cos\alpha}{f}P(v_0, v_1), \tag{6·7·6}$$

and

$$\frac{d\zeta_g}{dt} = \frac{d\theta_g}{dt} = \frac{\pi g\cos\alpha}{V}Q(v_0, v_1). \tag{6·7·7}$$

6·71. *Method of using the formulae*

Write

$$Z = Z_g + Z_T = (X+iY)\cos\alpha, \tag{6·71·1}$$

where Z_g is known from (6·7·2, 5) and Z_T from the initial tip-off conditions and (4·4·12), §§ 6·5, 6·6. The real part $X\cos\alpha$ is the *gravity drop*, i.e. the inclination of the trajectory to the horizontal at any instant during burning is

$$\alpha - X\cos\alpha.$$

The trajectory also bends to the right by an amount $-Y\cos\alpha$ in the plane of fire. This is equivalent to an amount $-Y$ to the right in the horizontal plane. For unrotated rockets $Y = 0$. For rotated rockets with $\gamma>0$, Y_g is usually negative or positive according as $\mathbf{M}_1$ is destabilizing or stabilizing, Y_T positive and $Y = Y_T + Y_g$ positive.

The drift at the point of graze may be estimated as follows. Suppose that the rocket is, at the instant of burnt, at a point Q in space. Let B (see figure 5) be the projection of Q upon the horizontal plane through the projector P, and let G be the point of graze in this plane. Let PF be the line of fire, and FB the projection of the tangent to the trajectory at Q on the horizontal plane, F being the point of intersection of the two lines. Let H, K and L be points on PF, FB and PF respectively such that the angles $\angle BHP$, $\angle GKB$ and $\angle GLP$ are right angles. Write

$$HB = D_1, \quad KG = D_2, \quad LG = D,$$
$$PH = R_1, \quad BK = R_2, \quad PL = R,$$
$$\angle LPG = \beta, \quad \angle LFB = \alpha_1.$$

The distances D_1, D_2 and D are taken positive when the points B, G and G are to the *right* of H, K and L respectively.

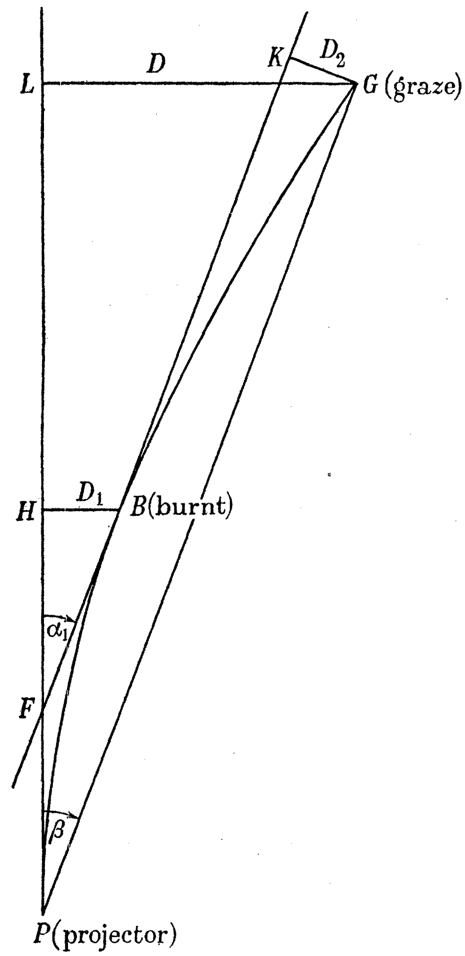

FIGURE 5. Drift in the horizontal plane during and after burning.

After burnt the motion of the rocket is similar to that of a shell, and a trajectory starting from Q with the initial velocity V_1, Q.E. $\alpha - X\cos\alpha$, and height $H_1 = \frac{1}{2}V_1^2 \sin\alpha/f$, may be calculated by the usual methods.† If drift is not allowed for, the point of graze in the horizontal plane is K, and the horizontal range will be R_2. The drift KG may then be cal-

† For this part of the trajectory infinite stability ($n = \infty$) may be assumed except for very large spins when overstability and yawing at the vertex may occur and so reduce the range.

culated from the trajectory data by means of Majevski's formula (see, for example, the *Text-book of anti-aircraft gunnery* (1925, p. 629) or Fowler *et al.* (1920, p. 358)).

Thus we know the quantities R_2 and D_2 from this trajectory, $\alpha_1 = -Y$ from (6·71·1) and (6·7·2), and

$$R_1 \doteqdot \tfrac{1}{2} V_1^2 \cos\alpha/f.$$

It is not necessary to have a precise estimate of D_1, the drift at burnt, since it is small in comparison with the other quantities concerned, its magnitude being of the order of $-\tfrac{1}{2}R_1 Y$.

The range to graze along the line of fire is then

$$R = R_1 + R_2$$

to a sufficient degree of accuracy, and the linear drift to the right is

$$\begin{aligned} D &= D_1 + D_2 + R_2\alpha_1 \\ &= D_2 - RY + (D_1 + R_1 Y) \\ &\doteqdot D_2 - RY, \end{aligned}$$

approximately. The angular drift at the point of graze, as viewed from the projector, is

$$\beta = \frac{D_2}{R} - Y \quad \text{to the } \textit{right},$$

and is therefore equal to the sum of the drift after burnt as viewed from the projector, and the component of the angular deviation at burnt in the horizontal plane.

6·8. Wind

In § 5·6 formulae were obtained for the angular deviation, yaw and rate of turn due to the wind. In the derivation of these expressions it was assumed that the projectile was perfectly launched. In actual fact part of the initial tip-off conditions will be due to the action of the wind, but the contribution of these parts to the motion during burning is very small and will be neglected. Accordingly, if $w_1(s)$ is the complex cross-wind (see (5·6·1)), we have, by (5·6·7, 8, 9),

$$\begin{aligned} Z = {} & \frac{1}{V}\left[w_1(s) - w_1(s_0)\, e^{i\beta\gamma(s-s_0)}\left\{\cos p(s-s_0) - i\frac{\beta\gamma}{p}\sin p(s-s_0)\right\}\right] \\ & - \frac{1}{V}\int_{s_0}^{s} w_1'(u)\, e^{i\beta\gamma(s-u)}\left\{\cos p(s-u) - i\frac{\beta\gamma}{p}\sin p(s-u)\right\} du \\ & + w_1(s_0)\frac{\nu^2}{p}\int_{s_0}^{s} e^{i\beta\gamma(u-s_0)}\sin p(u-s_0)\frac{du}{V_u} + \frac{\nu^2}{p}\int_{s_0}^{s}\frac{du}{V_u}\int_{s_0}^{u} w_1'(v)\, e^{i\beta\gamma(u-v)}\sin p(u-v)\, dv, \end{aligned} \qquad (6·8·1)$$

$$\begin{aligned} \Xi = {} & \frac{1}{V}\Big[w_1(s_0)\, e^{i\beta\gamma(s-s_0)}\left\{\cos p(s-s_0) - i\frac{\beta\gamma}{p}\sin p(s-s_0)\right\} - w_1(s) \\ & + \int_{s_0}^{s} w_1'(u)\, e^{i\beta\gamma(s-u)}\left\{\cos p(s-u) - i\frac{\beta\gamma}{p}\sin p(s-u)\right\} du\Big], \end{aligned} \qquad (6·8·2)$$

and

$$\frac{d\zeta}{dt} = w_1(s_0)\frac{\nu^2}{p}\, e^{i\beta\gamma(s-s_0)}\sin p(s-s_0) + \frac{\nu^2}{p}\int_{s_0}^{s} w_1'(u)\, e^{i\beta\gamma(s-u)}\sin p(s-u)\, du. \qquad (6·8·3)$$

6·81. *The wind structure*

We now assume a particular form for the wind gradient, namely

$$w_1(s) = w_1(s_0)\left\{1+\epsilon\log\frac{V}{V_0}\right\}. \tag{6·81·1}$$

This is similar to a form proposed by Sutton† (1936), namely,

$$w_1(s) = w_1(s_0)\left\{1+\frac{\log_{10} h/h_0}{\log_{10}\alpha}\right\}, \tag{6·81·2}$$

where h is the height and the constant α is a function of the temperature lapse rate, of the local topography, of h_0 and possibly of the wind speed $w_1(s_0)$. If we take a constant acceleration throughout burning ($V_{00} = 0$), then (6·81·1) and (6·81·2) agree with

$$\epsilon = \frac{2}{\log_e \alpha}, \tag{6·81·3}$$

since

$$h \doteqdot s\sin\alpha = V^2\sin\alpha/2f,$$

approximately, provided that the rear of the projectile is not too far off the ground at the time $t = 0$.

From an extensive survey of gun sites in the London area it was found that (6·81·2) gave a very good representation of the wind profile over a fairly wide range of heights.

The quantity ϵ in (6·81·1) is assumed to be constant. It is a pure number and will normally be real as in (6·81·3), but may be taken to be complex in order to cater for a wind varying in direction as well as in velocity. For a constant wind $\epsilon = 0$ of course. The relation between w_1 and the cross and line winds is given by (5·6·1).

From (6·81·1) we have

$$w_1'(s) = \epsilon w_1(s_0)\frac{f}{V^2}. \tag{6·81·4}$$

6·82. *Notation*

In order to evaluate the integrals (6·8·1, 2, 3) we introduce the following functions:

$$D^*(u) = B^*(u)+iA^*(u) = \int_u^\infty D(x)\frac{dx}{x}, \tag{6·82·1}$$

$$e(u) = \int_u^\infty e^{ix}\frac{dx}{x} = -\mathrm{Ci}\,x - i\,\mathrm{si}\,x = -\mathrm{Ci}\,x + i\{\tfrac{1}{2}\pi - \mathrm{Si}\,x\}, \tag{6·82·2}$$

$$e_k = e(\tfrac{1}{2}\pi w_{0k}^2) - e(\tfrac{1}{2}\pi w_{1k}^2). \tag{6·82·3}$$

The functions $A^*(u)$ and $B^*(u)$ may be calculated for small u from the formulae‡

$$A^*(u) = \tfrac{1}{8}\pi - \tfrac{1}{4}\{\mathrm{Ci}\,(\tfrac{1}{2}\pi u^2) - \mathrm{Si}\,(\tfrac{1}{2}\pi u^2)\} - \frac{\pi u^3}{1.3^2} + \frac{\pi^3 u^7}{1.3.5.7^2} - \frac{\pi^5 u^{11}}{1.3.5.7.9.11^2} + \ldots, \tag{6·82·4}$$

$$B^*(u) = -\tfrac{1}{8}\pi - \tfrac{1}{4}\{\mathrm{Ci}\,(\tfrac{1}{2}\pi u^2) + \mathrm{Si}\,(\tfrac{1}{2}\pi u^2)\} + u - \frac{\pi^2 u^5}{1.3.5^2} + \frac{\pi^4 u^9}{1.3.5.7.9^2} - \ldots. \tag{6·82·5}$$

† Sutton actually suggests the form

$$w_1(s) = w_1(s_0)\log_{10}\left(\frac{\alpha h}{h_0}+1\right)\Big/\log_{10}(\alpha+1)$$

which reduces to (6·81·2) when α is large.

‡ For tables of Ci x and Si x see the *Table of sine, cosine and exponential integrals*, published by the Work Projects Administration (1940).

For large u the following asymptotic expansions hold:

$$A^*(u) = \frac{1}{\pi u} - \frac{1.3}{5\pi^3 u^5} + \frac{1.3.5.7}{9\pi^5 u^9} - \ldots, \tag{6·82·6}$$

$$B^*(u) = \frac{1}{3\pi^2 u^3} - \frac{1.3.5}{7\pi^4 u^7} + \frac{1.3.5.7.9}{11\pi^6 u^{11}} - \ldots. \tag{6·82·7}$$

Tables of the functions $A^*(u)$ and $B^*(u)$ at intervals of 0·1 from 0·1 to 5·0 are given in Appendix A.

6·83. *Evaluation of the deviation, etc.*

We have, from (6·8·1, 2, 3), (6·81·1, 4) and (6·82·1, 2, 3),

$$Z = Z^{(1)} - Z^{(2)}, \tag{6·83·1}$$

where
$$Z^{(1)} = -\frac{\beta\gamma - p}{2pV} w_1(s_0) \left\{1 + i\pi w_{11} E_1 + \epsilon\left[\log\frac{V}{V_0} + i\pi w_{11}\{D^*(w_{01}) - D^*(w_{11})\} - \tfrac{1}{2} i\pi w_{11}\, e^{\frac{1}{2}\pi i w_{11}^2} \bar{e}_1 \left\{D(w_{11}) - \frac{i}{\pi w_{11}}\right\}\right]\right\} \tag{6·83·2}$$

and
$$Z^{(2)} = -\frac{\beta\gamma + p}{2pV} w_1(s_0) \left\{1 + i\pi w_{12} E_2 + \epsilon\left[\log\frac{V}{V_0} + i\pi w_{12}\{D^*(w_{02}) - D^*(w_{12})\} - \tfrac{1}{2} i\pi w_{12}\, e^{\frac{1}{2}\pi i w_{12}^2} \bar{e}_2 \left\{D(w_{12}) - \frac{i}{\pi w_{12}}\right\}\right]\right\}. \tag{6·83·3}$$

Also
$$\Xi = \frac{\beta\gamma - p}{2pV} w_1(s_0) \left\{1 - e^{i\psi_1} + \epsilon\left[\log\frac{V}{V_0} - \tfrac{1}{2} e^{i\psi_1} \bar{e}_1\right]\right\} - \frac{\beta\gamma + p}{2pV} w_1(s_0) \left\{1 - e^{i\psi_2} + \epsilon\left[\log\frac{V}{V_0} - \tfrac{1}{2} e^{i\psi_2} \bar{e}_1\right]\right\}, \tag{6·83·4}$$

and
$$\frac{d\zeta}{dt} = -i\frac{v^2}{2p} w_1(s_0)\, \{e^{i\psi_1} - e^{i\psi_2} + \tfrac{1}{2}\epsilon(e^{i\psi_1}\bar{e}_1 - e^{i\psi_2}\bar{e}_2)\}. \tag{6·83·5}$$

These formulae hold when $0 < p \leqslant \beta\gamma$. The corresponding formulae for a stabilizing lift moment ($p > \beta\gamma$) may be written down from (6·83·1 to 6·83·5) with the help of the table in § 6·4 and the following transformations. When

$$w_{j2}^2 \quad \text{becomes} \quad -w_{j2}^2,$$

$$w_{12} D^*(w_{j2}) \quad \text{becomes} \quad -w_{12} \bar{D}^*(w_{j2})$$

and
$$\bar{e}_2 \quad \text{becomes} \quad e_2.$$

The components of the angular deviation in the vertical plane and in the plane of fire may be found from the formulae for Z as follows. Write

$$Z = (X + iY)\, w_1(s_0). \tag{6·83·6}$$

Then the angular deviation in the vertical plane is, by (5·6·1) and (6·83·6),

$$Xw_F \sin\alpha + Yw_L \quad \textit{downwards}.$$

The lateral angular deviation is, similarly,

$$Xw_L - Yw_F \sin\alpha \quad \text{to the } \textit{right}$$

in the plane of fire. This is equivalent to a deviation

$$Xw_L \sec\alpha - Yw_F \tan\alpha \quad \text{to the } \textit{right}$$

in the horizontal plane.

6·84. *Unrotated rocket*

When there is no axial spin ($Y = 0$) we obtain

$$Z = -w_1(s_0)\sqrt{\left(\frac{n\pi}{f}\right)}\{E(v_0, v_1) + \epsilon K(v_0, v_1)\}, \tag{6·84·1}$$

where

$$\begin{aligned} K(v_0, v_1) = {} & A^*(v_0) - A^*(v_1) - \frac{1}{\pi v_1}\log\frac{v_1}{v_0} \\ & + \tfrac{1}{2}\{\mathrm{Ci}\,(\tfrac{1}{2}\pi v_0^2) - \mathrm{Ci}\,(\tfrac{1}{2}\pi v_1^2)\}\{B(v_1)\sin\tfrac{1}{2}\pi v_1^2 - A_1(v_1)\cos\tfrac{1}{2}\pi v_1^2\} \\ & - \tfrac{1}{2}\{\mathrm{Si}\,(\tfrac{1}{2}\pi v_0^2) - \mathrm{Si}\,(\tfrac{1}{2}\pi v_1^2)\}\{B(v_1)\cos\tfrac{1}{2}\pi v_1^2 + A_1(v_1)\sin\tfrac{1}{2}\pi v_1^2\}. \end{aligned} \tag{6·84·2}$$

Also

$$\begin{aligned} \Xi = -\frac{w_1(s_0)}{V}\Big[& 1 - \cos\tfrac{1}{2}\pi(v_1^2 - v_0^2) + \epsilon\log\frac{v_1}{v_0} + \tfrac{1}{2}\epsilon\{\mathrm{Ci}\,(\tfrac{1}{2}\pi v_0^2) - \mathrm{Ci}\,(\tfrac{1}{2}\pi v_1^2)\}\cos\tfrac{1}{2}\pi v_1^2 \\ & + \tfrac{1}{2}\epsilon\{\mathrm{Si}\,(\tfrac{1}{2}\pi v_0^2) - \mathrm{Si}\,(\tfrac{1}{2}\pi v_1^2)\}\sin\tfrac{1}{2}\pi v_1^2\Big], \end{aligned} \tag{6·84·3}$$

and

$$\begin{aligned} d\zeta/dt = -nw_1(s_0)[& \sin\tfrac{1}{2}\pi(v_1^2 - v_0^2) - \tfrac{1}{2}\epsilon\{\mathrm{Ci}\,(\tfrac{1}{2}\pi v_0^2) - \mathrm{Ci}\,(\tfrac{1}{2}\pi v_1^2)\}\sin\tfrac{1}{2}\pi v_1^2 \\ & + \tfrac{1}{2}\epsilon\{\mathrm{Si}\,(\tfrac{1}{2}\pi v_0^2) - \mathrm{Si}\,(\tfrac{1}{2}\pi v_1^2)\}\cos\tfrac{1}{2}\pi v_1^2]. \end{aligned} \tag{6·84·4}$$

In these formulae $\qquad w_1 = w_F \sin\alpha - iw_L,$

where w_F and w_L are the wind components along the line of fire and from left to right. It follows that the angular deviation in the vertical plane due to a following wind w_F is

$$-w_F(s_0)\sin\alpha\sqrt{\left(\frac{n\pi}{f}\right)}\{E(v_0, v_1) + \epsilon K(v_0, v_1)\} \tag{6·84·5}$$

downwards. Similarly, the angular deviation in the plane of fire due to a wind w_L from left to right is

$$-w_L(s_0)\sqrt{\left(\frac{n\pi}{f}\right)}\{E(v_0, v_1) + \epsilon K(v_0, v_1)\} \tag{6·84·6}$$

to the right.

6·85. *Method of using the formulae*

The expressions (6·83·1) and (6·84·1) for the angular deviation have been obtained under the assumption that the wind speed varies logarithmically with the velocity—and therefore with the height—according to the law (6·81·1). This includes the case of a wind which is constant in speed and direction at each part of the trajectory during burning, for $\epsilon = 0$.

In order to use the formulae it is necessary to know the wind speeds and directions at a series of heights and to fit a curve of the form (6·81·1) to these data in order to determine the appropriate parameters $w_1(s_0)$ and ϵ. For unrotated motion, however, it is possible to do without extensive wind measurements and to avoid curve fitting by making use of the concept of *equivalent height*. The angular deviation at burnt can then be determined from a single measurement of the wind components at this height. The equivalent height depends only upon the Q.E. and the design of the particular rocket considered, and is independent of wind speed and gradient. It is defined in the following manner.

Suppose that the angular deviation at burnt is

$$Z_1 = w_1(s_0)\,X_1 = w_1(s_0)\,(X_{11}+\epsilon X_{12}), \tag{6·85·1}$$

where X_{11}, X_{12} are real constants which are given by

$$X_{11} = -\sqrt{\left(\frac{n\pi}{f}\right)}E(v_0, v_1), \quad X_{12} = -\sqrt{\left(\frac{n\pi}{f}\right)}K(v_0, v_1). \tag{6·85·2}$$

Write

$$V_e = V_0 \exp\,(X_{12}/X_{11}), \tag{6·85·3}$$

and suppose that† $s = s_e$ is the distance travelled, and $h = h_e$ the height above the ground at the instant before burnt when $V = V_e$; h_e is the equivalent height, and it is clear that it depends only upon the Q.E., the initial height of the projector, and upon the constants X_{11}, X_{12}. If the acceleration is constant throughout the burning period and if the height of the centre of gravity at the instant $t = 0$ is h_{00}, then

$$h_e = h_{00}+s_e \sin\alpha = h_{00}+\frac{V_e^2}{2f}\sin\alpha. \tag{6·85·4}$$

It follows from (6·85·1, 3) and (6·81·1) that

$$Z_1 = w_1(s_e)\,X_{11}. \tag{6·85·5}$$

The quantity X_{11} is the *wind constant* for a constant wind ($\epsilon = 0$). The real and imaginary parts of $w_1(s_e)$, namely $w_F(s_e)\sin\alpha$ and $-w_L(s_e)$, which give the wind components perpendicular to the trajectory at the equivalent height h_e are called the *equivalent wind components*. Thus the angular deviation at burnt due to wind can be determined, when the wind constant X_{11} is known, from the wind-speed components at the equivalent height by the simple relation (6·85·5).

If the projectile is rotated ($\gamma > 0$) this method is no longer possible. For if we proceed along the same lines we find that there are two equivalent heights, corresponding to the two perpendicular planes through the trajectory, each of which is dependent on the wind components as well as upon the constants X_{11}, X_{12}, Y_{11}, Y_{12}.

6·9. The effect of tolerances upon the unrotated rocket

When the projectile has no axial spin the only tolerances which affect the motion are those due to a malalined thrust and a malalined exit-plane centre, namely α_R and α_N. The other four tolerances α_C, α_G, α_L and α_M only occur in the tolerance functions μ_1, μ_2 and μ_3 as coefficients of the spin r or of the rotational couple G_R. Thus we have, by (4·2·9, 10, 11),

$$\mu_1 = \alpha_R e^{i\phi_R}, \quad \mu_2 = \frac{ml}{A}\{\alpha_N e^{i\phi_N}-\alpha_R e^{i\phi_R}\}, \quad \mu_3 = 0. \tag{6·9·1}$$

When the projectile is perfectly launched the angular deviation, yaw, and rate of turn may be obtained from this with the help of the expressions (5·7·16 to 5·7·18) or (5·75·1 to 5·75·3) and (5·76·1 to 5·76·3), namely,

$$\Xi = x_1(\alpha_R e^{i\phi_R}) - x_2\left(\frac{ml}{A}\alpha_R e^{i\phi_R}\right) + x_2\left(\frac{ml}{A}\alpha_N e^{i\phi_N}\right), \tag{6·9·2}$$

$$\frac{d\zeta}{dt} = y_1(\alpha_R e^{i\phi_R}) - y_2\left(\frac{ml}{A}\alpha_R e^{i\phi_R}\right) + y_2\left(\frac{ml}{A}\alpha_N e^{i\phi_N}\right), \tag{6·9·3}$$

and

$$Z = z_1(\alpha_R e^{i\phi_R}) - z_2\left(\frac{ml}{A}\alpha_R e^{i\phi_R}\right) + z_2\left(\frac{ml}{A}\alpha_N e^{i\phi_N}\right), \tag{6·9·4}$$

† This is only possible if $V_e \leqslant V_1$, a condition which is always satisfied in practice.

if the magnitudes of the four angles α_R, ϕ_R, α_N, ϕ_N are known at each instant during burning. Measurements of lateral thrusts during static firings have shown that considerable fluctuations can occur in the values of α_R and ϕ_R during the burning period. The magnitude of α_N is usually much smaller than α_R, and it is probable that it and ϕ_N do not fluctuate so rapidly as α_R and ϕ_R.

There is, of course, no method of estimating exactly the magnitudes and variations of the four angles for a given rocket without firing it, so that the formulae (6·9·2 to 6·9·4) cannot be used in order to predict the exact trajectory of such a rocket except in special cases where large malalinements are deliberately built in. The chief value of the formulae lies in their use for predicting the dispersion to be expected from a homogeneous series of rockets fired under equal conditions. For this purpose an approximate estimate of the dispersion in angular deviation due to any particular tolerance may be obtained by substituting for $\alpha_P e^{i\phi_P}$ ($P = N$ or R) the appropriate value of the standard deviation of α_P, this being assumed to be constant throughout burning (see § 9·8). It is for this reason that it is of interest to know the deviation, yaw and rate of turn due to constant malalinements, and accordingly Assumption C 3 has been made. For the case of the unrotated rocket this assumption states that variations in

$$\alpha_R, \quad \phi_R, \quad \frac{ml}{A}\alpha_R, \quad \frac{ml}{A}\alpha_N, \quad \phi_N$$

may be neglected during burning, and the problem reduces to that of finding the values of the functions

$$x_1(1), \quad x_2(1), \quad y_1(1), \quad y_2(1), \quad z_1(1), \quad z_2(1).$$

Since $\gamma = 0$ we have, from (6·3·1, 2),

$$P(s) = \Lambda(s) = \sigma(s) = 0, \quad p = n, \tag{6·9·5}$$

and hence, by the formulae of § 5·7 and (6·4·13, 14, 23, 24),

$$x_1(1) = -\frac{1}{V}\int_{s_0}^{s} \cos n(s-u)\, dV_u = -P(v_0, v_1), \tag{6·9·6}$$

$$x_2(1) = \frac{1}{nV}\int_{s_0}^{s} \sin n(s-u)\, dV_u = \frac{1}{n}Q(v_0, v_1), \tag{6·9·7}$$

$$y_1(1) = n\int_{s_0}^{s} \sin n(s-u)\, dV_u = nVQ(v_0, v_1), \tag{6·9·8}$$

$$y_2(1) = \int_{s_0}^{s} \cos n(s-u)\, dV_u = VP(v_0, v_1), \tag{6·9·9}$$

$$z_1(1) = \frac{1}{V}\int_{s_0}^{s} \cos n(s-u)\, dV_u + n\int_{s_0}^{s}\frac{du}{V_u}\int_{s_0}^{u} \sin n(u-v)\, dV_v = H(v_0, v_1), \tag{6·9·10}$$

$$z_2(1) = -\frac{1}{nV}\int_{s_0}^{s} \sin n(s-u)\, dV_u + \int_{s_0}^{s}\frac{du}{V_u}\int_{s_0}^{u} \cos n(u-v)\, dV_v = \frac{1}{n}F(v_0, v_1). \tag{6·9·11}$$

In the following two subsections we consider the two tolerances α_R and α_N separately.

Tip-off. The expressions (6·9·2 to 6·9·4) hold when the projectile is perfectly launched. In general, however, the two malalinements α_R and α_N will each contribute to the tip-off conditions at launch, as explained in § 4·4. Thus the total angular deviation due to tolerances will take the form

$$Z = (Z_R + Z_{RT}) + (Z_N + Z_{NT}), \tag{6·9·12}$$

where the deviations Z_R and Z_N are those which are obtained when the projectile is perfectly launched, and Z_{RT} and Z_{NT} are the deviations due to the initial tip-off conditions produced by α_R and α_N respectively. When the initial values at launch of the angular deviation, yaw, and rate of turn due to each of α_R and α_N are known, the values of Z_{RT} and Z_{NT} can be evaluated with the help of (4·4·12) and §§ 6·5, 6·6. It is found, in nearly all practical cases, that Z_{RT} is much smaller than Z_R and Z_{NT} than Z_N. Similar considerations apply to the yaw and rate of turn.

6·91. *Malalined thrust*

We have, from (6·9·2 to 6·9·4) and (6·9·6 to 6·9·11),

$$Z_R = \alpha_R e^{i\phi_R}\left\{H(v_0, v_1) - \frac{ml}{An} F(v_0, v_1)\right\}, \tag{6·91·1}$$

$$\Xi_R = -\alpha_R e^{i\phi_R}\left\{P(v_0, v_1) + \frac{ml}{An} Q(v_0, v_1)\right\}, \tag{6·91·2}$$

and

$$\frac{d\zeta_R}{dt} = \alpha_R e^{i\phi_R} nV\left\{Q(v_0, v_1) - \frac{ml}{An} P(v_0, v_1)\right\}. \tag{6·91·3}$$

Since ml/An is usually large, the second term in each curly bracket will usually predominate over the first term. This means that the moment of the malalined thrust about the centre of gravity is of more importance than the actual lateral force due to the malalinement.

The factor $e^{i\phi_R}$ specifies the plane in which the motion takes place; e.g. if $\phi_R = 0$ it is the downward vertical plane.

6·92. *Malalined exit-plane centre*

We have, from (6·9·2 to 6·9·4) and (6·9·6 to 6·9·11),

$$Z_N = \alpha_N e^{i\phi_N} \frac{ml}{An} F(v_0, v_1), \tag{6·92·1}$$

$$\Xi_N = \alpha_N e^{i\phi_N} \frac{ml}{An} Q(v_0, v_1), \tag{6·92·2}$$

and

$$\frac{d\zeta_N}{dt} = \alpha_N e^{i\phi_N} \frac{mlV}{A} P(v_0, v_1). \tag{6·92·3}$$

The factor $e^{i\phi_N}$ specifies the plane in which the motion takes place; e.g. if $\phi_N = 0$ it is the downward vertical plane.

6·10. The effect of tolerances upon the rotated rocket

We examine here the effects of the six tolerances

$$\alpha_C, \quad \alpha_G, \quad \alpha_L, \quad \alpha_M, \quad \alpha_N, \quad \alpha_R$$

upon the motion of the rocket during the burning period when no other disturbing forces act. The angular deviations due to any one of these tolerances α_P, say, may be written in the form

$$Z_P + Z_{PT},$$

where Z_P is the angular deviation of the trajectory when perfect launch is assumed, and Z_{PT} is the angular deviation caused by those parts of the initial tip-off conditions which depend upon α_P. When the initial values at launch of the angular deviation, yaw and rate of turn due to the presence of the tolerance α_P are known, Z_{PT} can be calculated with the help of (4·4·12) and §§ 6·5, 6·6. It is the purpose of this section to derive formulae for the remaining part Z_P, and for the corresponding yaw Ξ_P and rate of turn $d\zeta_P/dt$.

The values of Ξ_P, $d\zeta_P/dt$ and Z_P for each tolerance can be found from (5·7·16 to 5·7·18) in terms of the functions x_ν, y_ν, z_ν ($\nu = 1, 2, 3$) which are defined by (5·7·6 to 5·7·15), when the magnitudes of α_P and ϕ_P, and therefore μ_1, μ_2 and μ_3, are known at each instant during burning. For normal service rockets it is probable that each α_P and ϕ_P fluctuates considerably during burning. No satisfactory method has so far been devised which determines all six tolerances for any given projectile (see § 9·8). In most cases it is only possible to state that for a homogeneous batch of ammunition the points whose co-ordinates are ($\alpha_P \cos\phi_P$, $\alpha_P \sin\phi_P$) are distributed randomly with a linear standard deviation σ_P, which is deduced by methods which are often empirical, and is assumed to be constant during burning. By substituting σ_P for $\alpha_P e^{i\phi_P}$ in the formula for Z_P, a rough estimate of the contribution of the particular tolerance α_P to the angular dispersion of the ammunitions may be obtained.† The quantities $\alpha_P e^{i\phi_P}$ need not, of course, all be independent of each other,‡ though it is likely that most of them are independent.

It is for this reason that it is useful to have formulae for Z_P, etc., for constant or nearly constant $\alpha_P e^{i\phi_P}$, and accordingly Assumption C 3 has been made. In addition to their use for estimating dispersions, these formulae can be employed to find the motion of special rockets which have had large known malalinements deliberately built in.

Assumption C 3 states that variations in the quantities $a_P(\nu)\,\alpha_P e^{i\phi_P}$, $b_P(\nu)\,\alpha_P e^{i\phi_P}$ ($\nu = 1, 2, 3$) defined by (3·63·1) may be neglected. Of these coefficients all are zero except the following:

$$b_C(2) = (1-2\beta)\frac{m(dA/dt)}{QWA}, \quad b_C(3) = -(1-2\beta), \tag{6·10·1}$$

$$b_G(2) = -\frac{mlq_1}{WA}, \tag{6·10·2}$$

$$a_L(2) = -i\frac{mG_R}{QWA}, \tag{6·10·3}$$

$$b_M(2) = \frac{mlq_1}{WA}, \tag{6·10·4}$$

$$b_N(1) = 2\frac{l}{W}, \quad a_N(2) = \frac{ml}{A}, \quad b_N(2) = -\frac{ml^2}{AW}, \tag{6·10·5}$$

$$a_R(1) = 1, \quad a_R(2) = -\frac{ml}{A}. \tag{6·10·6}$$

Since $a_P(3) = 0$ for every P it is clear from (5·7·16 to 5·7·18) that we have to evaluate fifteen functions, namely,

$$x_1(1), \quad x_2(1), \quad x_1(ir), \quad x_2(ir), \quad x_3(ir),$$

† This estimate can, of course, only be very approximate unless some account is taken of the variations in ϕ_P and α_P during burning. In the case of rotated motion there is the additional possibility that larger deviations may occasionally be caused when the fluctuations of $\alpha_P\, e^{i\phi_P}$ resonate with the spin factor $e^{i\gamma s}$.

‡ E.g. it is probable that there is some correlation between $\alpha_M\, e^{i\phi_M}$ and $\alpha_N\, e^{i\phi_N}$.

and the corresponding y and z functions. This is done in the succeeding subsection. In certain cases it is necessary to approximate in order to obtain expressions suitable for numerical calculation. These approximations are valid provided that

$$\left.\begin{array}{l}\text{(i) } \lambda_1, \lambda_2 \text{ are small in comparison with unity,}\\ \text{(ii) } \gamma s_0 = \tfrac{1}{2}\pi G_0^2 \text{ is not small, say greater than 4.}\end{array}\right\} \tag{6·10·7}$$

Further approximations are possible at points near burnt where γs is large.

6·101. *Evaluation of the functions*

Let
$$R_k = \frac{1}{G_{1k}}\{e^{i\psi_k}D(G_{0k}) - e^{i\phi}D(G_{1k})\}. \tag{6·101·1}$$

We consider first the five x functions. We obtain from Assumptions C 1 to 3 and the definitions in §§ 6·3, 6·4 and (6·101·1) the following expressions which hold when the lift moment is destabilizing $(0 < p \leqslant \beta\gamma)$:

$$x_1(1) = \frac{e^{i\sigma_0}}{2p}\{(\beta\gamma - p)R_1 - (\beta\gamma + p)R_2\}, \tag{6·101·2}$$

$$x_2(1) = -i\frac{e^{i\sigma_0}}{2p}\{R_1 - R_2\}, \tag{6·101·3}$$

$$x_1(ir) = -\frac{fe^{i\sigma_0}}{2pV}\left\{\frac{\beta\gamma - p}{1-\lambda_1}(e^{i\psi_1} - e^{i\phi}) - \frac{\beta\gamma + p}{1-\lambda_2}(e^{i\psi_2} - e^{i\phi})\right\}, \tag{6·101·4}$$

$$x_2(ir) = i\frac{fe^{i\sigma_0}}{2pV}\left\{\frac{1}{1-\lambda_1}(e^{i\psi_1} - e^{i\phi}) - \frac{1}{1-\lambda_2}(e^{i\psi_2} - e^{i\phi})\right\}, \tag{6·101·5}$$

$$x_3(ir) = -\frac{\gamma e^{i\sigma_0}}{2pV}\left\{\frac{1}{1-\lambda_1}(V_0 e^{i\psi_1} - \lambda_1 V e^{i\phi} + \lambda_1 V R_1) - \frac{1}{1-\lambda_2}(V_0 e^{i\psi_2} - \lambda_2 V e^{i\phi} + \lambda_2 V R_2)\right\}. \tag{6·101·6}$$

Similarly we have
$$y_1(1) = i\frac{\nu^2 V e^{i\sigma_0}}{2p}\{R_1 - R_2\}, \tag{6·101·7}$$

$$y_2(1) = \frac{Ve^{i\sigma_0}}{2p}\{(\beta\gamma + p)R_1 - (\beta\gamma - p)R_2\}, \tag{6·101·8}$$

$$y_1(ir) = -i\frac{\nu^2 fe^{i\sigma_0}}{2p}\left\{\frac{1}{1-\lambda_1}(e^{i\psi_1} - e^{i\phi}) - \frac{1}{1-\lambda_2}(e^{i\psi_2} - e^{i\phi})\right\}, \tag{6·101·9}$$

$$y(ir) = -\frac{fe^{i\sigma_0}}{2p}\left\{\frac{\beta\gamma + p}{1-\lambda_1}(e^{i\psi_1} - e^{i\phi}) - \frac{\beta\gamma - p}{1-\lambda_2}(e^{i\psi_2} - e^{i\phi})\right\}, \tag{6·101·10}$$

$$y_3(ir) = i\frac{\gamma^2 e^{i\sigma_0}}{2p}\left\{\frac{\lambda_1}{1-\lambda_1}(Ve^{i\phi} - V_0 e^{i\psi_1} - \lambda_1 V R_1) - \frac{\lambda_2}{1-\lambda_2}(Ve^{i\phi} - V_0 e^{i\psi_2} - \lambda_2 V R_2)\right\}. \tag{6·101·11}$$

It will be observed that each of the ten functions (6·101·2 to 6·101·11) is expressed as a difference of two parts in a manner similar to that employed in §§ 6·5 to 6·8. In these sections the two parts corresponded to the two modes of precession of wave-lengths $2\pi/\gamma\lambda_1$ and $2\pi/\gamma\lambda_2$. In the present case, however, each part contains terms in $e^{i\phi}$ as well as in $e^{i\psi_1}$ and $e^{i\psi_2}$. This shows that the motion is compounded of three precessional motions instead of the usual two. The presence of the high-frequency terms in $e^{i\phi}$ could, of course, have been foreseen, since the disturbing forces due to the tolerances do not remain constant in direction but are carried round with the rocket as it rotates. It will usually be convenient to collect the various terms in $e^{i\phi}$ together when calculating the functions (6·101·2 to 6·101·11). At points of the trajectory not too near launch it is possible to approximate to (6·101·6, 11) by omitting the terms in R_1, R_2 and to (6·101·2, 3, 7, 8) by writing

$$\frac{1}{G_{1k}}\,e^{i\psi_k}D(G_{0k}) \quad \text{in place of } R_k.$$

The evaluation of the five z functions is not so easy. It is necessary, in three cases, to approximate to the function

$$I_k = \gamma\lambda_k\,e^{-i\gamma s_0}\int_{s_0}^{s} e^{i\gamma\lambda_k u}\frac{du}{V_u}\int_{s_0}^{u} e^{i\gamma(1-\lambda_k)v}\,dV_v.$$

This can be done by replacing the double integral by

$$\int_{s_0}^{s}\int_{s_0}^{\infty} - \int_{s_0}^{s}\int_{u}^{\infty}.$$

The first part contributes

$$\pi\frac{w_{1k}}{G_{1k}}D(G_{0k})\,\{D(w_{0k}) - e^{i\psi_k}D(w_{1k})\};$$

the second part may be expanded in an asymptotic series, by repeated integration by parts, the first term being

$$\frac{\lambda_k}{\pi}\left\{\frac{1}{G_{01}^2} - \frac{e^{i\phi}}{G_{11}^2}\right\}.$$

Thus we have, under assumption (6·10·7),

$$I_k \doteqdot \pi\frac{w_{1k}}{G_{1k}}D(G_{0k})\,\{D(w_{0k}) - e^{i\psi_k}D(w_{1k})\} + \frac{\lambda_k}{\pi}\left\{\frac{1}{G_{01}^2} - \frac{e^{i\phi}}{G_{11}^2}\right\}. \tag{6·101·12}$$

It follows from (5·7·13 to 5·7·16) and (6·101·12) that

$$z_1(1) \doteqdot i\frac{e^{i\sigma_0}}{2p}\left[(\beta\gamma - p)\left\{\frac{\pi w_{11}}{G_{11}}D(G_{01})\,E_1 + \frac{\lambda_1}{\pi G_{01}^2} - i\frac{e^{i\phi}}{G_{11}}\left(D(G_{11}) - i\frac{\lambda_1}{\pi G_{11}}\right)\right\}\right.$$
$$\left. - (\beta\gamma + p)\left\{\frac{\pi w_{12}}{G_{12}}D(G_{02})\,E_2 + \frac{\lambda_2}{\pi G_{02}^2} - i\frac{e^{i\phi}}{G_{12}}\left(D(G_{12}) - i\frac{\lambda_2}{\pi G_{12}}\right)\right\}\right], \tag{6·101·13}$$

$$z_2(1) \doteqdot \frac{e^{i\sigma_0}}{2p}\left[\left\{\frac{\pi w_{11}}{G_{11}}D(G_{01})\,E_1 + \frac{\lambda_1}{\pi G_{01}^2} - i\frac{e^{i\phi}}{G_{11}}\left(D(G_{11}) - i\frac{\lambda_1}{\pi G_{11}}\right)\right\}\right.$$
$$\left. - \left\{\frac{\pi w_{12}}{G_{12}}D(G_{02})\,E_2 + \frac{\lambda_2}{\pi G_{02}^2} - i\frac{e^{i\phi}}{G_{12}}\left(D(G_{12}) - i\frac{\lambda_2}{\pi G_{12}}\right)\right\}\right]. \tag{6·101·14}$$

When G_{1k} is large, e.g. at burnt, we can write

$$D(G_{1k}) = \frac{i}{\pi G_{1k}},$$

so that in formulae (6·101·13, 14) we may replace

$$-i\frac{e^{i\phi}}{G_{1k}}\left(D(G_{1k}) - i\frac{\lambda k}{\pi G_{1k}}\right) \quad \text{by} \quad \frac{e^{i\phi}}{\pi G_1^2}.$$

Also we have

$$z_1(ir) = -i\frac{v^2 V e^{i\sigma_0}}{2p}\left[\frac{1}{1-\lambda_1}\left\{\frac{E_1}{w_{11}} - \frac{1}{G_1}\left(D(G_0) - e^{i\phi}D(G_1) + \frac{iG_1}{\pi w_{11}^2}e^{i\phi}\right)\right\}\right.$$
$$\left. - \frac{1}{1-\lambda_2}\left\{\frac{E_2}{w_{12}} - \frac{1}{G_1}\left(D(G_0) - e^{i\phi}D(G_1) + \frac{iG_1}{\pi w_{12}^2}e^{i\phi}\right)\right\}\right], \tag{6·101·15}$$

$$\doteqdot -i\frac{v^2 V e^{i\sigma_0}}{2p}\left[\frac{E_1 - (i\,e^{i\phi}/\pi w_{11})}{(1-\lambda_1)\,w_{11}} - \frac{E_2 - (i\,e^{i\phi}/\pi w_{12})}{(1-\lambda_2)\,w_{12}}\right], \tag{6·101·16}$$

$$z_2(ir) = -\frac{V\,e^{i\sigma_0}}{2p}\left[\frac{\beta\gamma+p}{1-\lambda_1}\left\{\frac{E_1}{w_{11}}-\frac{1}{G_1}\left(D(G_0)-e^{i\phi}D(G_1)+\frac{iG_1}{\pi w_{11}^2}e^{i\phi}\right)\right\}\right.$$

$$\left.-\frac{\beta\gamma-p}{1-\lambda_2}\left\{\frac{E_2}{w_{12}}-\frac{1}{G_1}\left(D(G_0)-e^{i\phi}D(G_1)+\frac{iG_1}{\pi w_{12}^2}e^{i\phi}\right)\right\}\right], \quad (6\cdot101\cdot17)$$

$$\doteqdot -\frac{V\,e^{i\sigma_0}}{2p}\left[\frac{\beta\gamma+p}{w_{11}(1-\lambda_1)}\left\{E_1-\frac{i\,e^{i\phi}}{\pi w_{11}}\right\}-\frac{\beta\gamma-p}{w_{12}(1-\lambda_2)}\left\{E_2-\frac{i\,e^{i\phi}}{\pi w_{12}}\right\}\right], \quad (6\cdot101\cdot18)$$

and

$$z_3(ir) = z_{31}(ir)-z_{32}(ir), \quad (6\cdot101\cdot19)$$

where

$$z_{31}(ir) \doteqdot -i\frac{\gamma\,e^{i\sigma_0}}{2p(1-\lambda_1)}\left[\pi w_{01}E_1\left\{1+\lambda_1\frac{D(G_{01})}{G_{01}}\right\}-i\lambda_1\left\{1+\frac{i\lambda_1}{\pi G_{01}^2}\right\}-i\frac{\lambda_1 e^{i\phi}}{G_{11}}\left\{D(G_{11})-\frac{i\lambda_1}{\pi G_{11}}\right\}\right], \quad (6\cdot101\cdot20)$$

and

$$z_{32}(ir) \doteqdot -i\frac{\gamma\,e^{i\sigma_0}}{2p(1-\lambda_2)}\left[\pi w_{02}E_2\left\{1+\lambda_2\frac{D(G_{02})}{G_{02}}\right\}-i\lambda_2\left\{1+\frac{i\lambda_2}{\pi G_{02}^2}\right\}-i\frac{\lambda_2 e^{i\phi}}{G_{12}}\left\{D(G_{12})-\frac{i\lambda_2}{\pi G_{12}}\right\}\right]. \quad (6\cdot101\cdot21)$$

Hence, approximating further, we obtain

$$z_3(ir) = -i\frac{\gamma\,e^{i\sigma_0}}{2p}\left[\frac{\pi w_{01}E_1-i\lambda_1}{1-\lambda_1}-\frac{\pi w_{02}E_2-i\lambda_2}{1-\lambda_2}\right]. \quad (6\cdot101\cdot22)$$

These formulae apply to the case of a destabilizing lift moment. When this moment is stabilizing ($p>\beta\gamma$) the changes necessary may be found from the table in § 6·4.

In the following six subsections we obtain formulae for the yaw, cross-spin and angular deviation due to each of the six tolerances separately. These results are obtained by inserting the appropriate values of μ_1, μ_2 and μ_3, as given by (4·2·9 to 4·2·11), in the equations (5·7·16 to 5·7·18), and by using the formulae for the x, y and z functions which have just been derived.

6·102. *Displacement of the principal longitudinal axis of inertia*

We have

$$\Xi_C = -\alpha_C e^{i\phi_C}(1-2\beta)\left\{x_3(ir)-\frac{m(dA/dt)}{QWA}x_2(ir)\right\}, \quad (6\cdot102\cdot1)$$

$$\frac{d\zeta_C}{dt} = -\alpha_C e^{i\phi_C}(1-2\beta)\left\{y_3(ir)-\frac{m(dA/dt)}{QWA}y_2(ir)\right\}, \quad (6\cdot102\cdot2)$$

and

$$Z_C = -\alpha_C e^{i\phi_C}(1-2\beta)\left\{z_3(ir)-\frac{m(dA/dt)}{QWA}z_2(ir)\right\}, \quad (6\cdot102\cdot3)$$

where the x, y and z functions are given by (6·101·5 to 6·101·6, 10 to 11, 17 to 22).

The contributions from the terms in $m(dA/dt)/QWA$ are usually much smaller than those from the other terms. By (9·2·16) we have

$$-\frac{m(dA/dt)}{QWA} = \frac{m\kappa_A^2+q^2m_{00}^2/m}{WA}.$$

6·103. *Malalined charge centre with respect to exit plane and centre of gravity*

We have $$\Xi_G = -\alpha_G \, e^{i\phi_G} \frac{mlq_1}{AW} x_2(ir), \qquad (6·103·1)$$

$$\frac{d\zeta_G}{dt} = -\alpha_G \, e^{i\phi_G} \frac{mlq_1}{AW} y_2(ir), \qquad (6·103·2)$$

and $$Z_G = -\alpha_G \, e^{i\phi_G} \frac{mlq_1}{AW} z_2(ir), \qquad (6·103·3)$$

where the x, y and z functions are given by (6·101·5, 10, 17, 18).

6·104. *Malalined axial torque*

We have $$\Xi_L = -i\alpha_L \, e^{i\phi_L} \frac{mG_R}{QWA} x_2(1), \qquad (6·104·1)$$

$$\frac{d\zeta_L}{dt} = -i\alpha_L \, e^{i\phi_L} \frac{mG_R}{QWA} y_2(1), \qquad (6·104·2)$$

and $$Z_L = -i\alpha_L \, e^{i\phi_L} \frac{mG_R}{QWA} z_2(1), \qquad (6·104·3)$$

where the x, y and z functions are given by (6·101·3, 8, 14).

The quantity mG_R/QWA may be written, by (9·4·5) below,

$$\frac{mG_R}{QWA} = \frac{ma_e}{A} \tan \Delta. \qquad (6·104·4)$$

6·105. *Displaced thrust-application point*

We have $$\Xi_M = \alpha_M \, e^{i\phi_M} \frac{mlq_1}{WA} x_2(ir), \qquad (6·105·1)$$

$$\frac{d\zeta_M}{dt} = \alpha_M \, e^{i\phi_M} \frac{mlq_1}{WA} y_2(ir), \qquad (6·105·2)$$

and $$Z_M = \alpha_M \, e^{i\phi_M} \frac{mlq_1}{WA} z_2(ir), \qquad (6·105·3)$$

where the x, y and z functions are given by (6·101·5, 10, 17, 18).

6·106. *Malalined exit-plane centre*

We have $$\Xi_N = \alpha_N \, e^{i\phi_N} \left\{ \frac{ml}{A} x_2(1) + \frac{2l}{W} x_1(ir) - \frac{ml^2}{AW} x_2(ir) \right\}, \qquad (6·106·1)$$

$$\frac{d\zeta_N}{dt} = \alpha_N \, e^{i\phi_N} \left\{ \frac{ml}{A} y_2(1) + \frac{2l}{W} y_1(ir) - \frac{ml^2}{AW} y_2(ir) \right\}, \qquad (6·106·2)$$

and $$Z_N = \alpha_N \, e^{i\phi_N} \left\{ \frac{ml}{A} z_2(1) + \frac{2l}{W} z_1(ir) - \frac{ml^2}{AW} z_2(ir) \right\}, \qquad (6·106·3)$$

where the x, y and z functions are given by (6·101·3 to 6·101·5, 8 to 10, 14 to 18). The first term in each curly bracket will usually predominate.

6·107. *Malalined thrust*

We have
$$\Xi_R = \alpha_R e^{i\phi_R}\left\{x_1(1)-\frac{ml}{A}x_2(1)\right\}, \qquad (6\cdot 107\cdot 1)$$

$$\frac{d\zeta_R}{dt} = \alpha_R e^{i\phi_R}\left\{y_1(1)-\frac{ml}{A}y_2(1)\right\}, \qquad (6\cdot 107\cdot 2)$$

and
$$Z_R = \alpha_R e^{i\phi_R}\left\{z_1(1)-\frac{ml}{A}z_2(1)\right\}, \qquad (6\cdot 107\cdot 3)$$

where the x, y and z functions are given by (6·101·2, 3, 7, 8, 13, 14). The second term in each curly bracket will usually predominate.

7. SOLUTIONS FOR OTHER FORMS OF SPIN

7·1. General

We consider in this section a number of other particular cases in which it is possible to obtain approximate solutions of the general equations in forms more or less suitable for numerical computation. The assumptions which we shall make are the same as those made in § 6, except that we no longer assume that the spin is proportional to the velocity. Accordingly we neglect all Magnus and damping effects and assume that the acceleration is constant between launch and burnt. This means that the lift moment $\mathbf{M}_1$, the force of gravity and the jet forces are the only disturbing factors taken into account. Further assumptions occurring in particular cases are listed as they arise.

7·2. Large constant spin and small or zero lift moment

In this section we obtain solutions of the equations of motion when the axial spin r is constant. These solutions are valid when the contribution of the lift moment to the equations can be neglected, i.e. when the terms in n^2 can be omitted. This will be so, as we shall see, when (i) $n^2 = 0$, i.e. for neutral stability, or (ii) when the spin is sufficiently large and n^2 sufficiently small, i.e. when the projectile is 'overspun'. It is accordingly to be expected that the motion will show the well-known symptoms of overspinning, i.e. that the axis of the projectile will tend to remain pointing in the initial direction so that the yaw will build up as the trajectory bends down under gravity.

The main equation governing the motion of the projectile about its centre of gravity is (4·3·8). The parameter n^2 occurs in this equation in the coefficient $G(s)$ of H. And, by (4·2·17 to 4·2·19), we have, since Magnus and damping terms are neglected,

$$G(s) = n^2+\beta^2r^2/V^2-ir\beta f/V^3.$$

It follows that we are justified in neglecting n^2 when

$$\frac{n^2V^2}{\beta^2r^2}$$

is small. This then may be taken to be the precise form of condition (ii) above.

We therefore suppose from now on that n^2 can be neglected. It is convenient to use equations (4·3·3, 4) rather than (4·3·8, 9). The former become, in view of our assumptions,

$$V\frac{dZ}{dt}=f(\zeta-Z+\mu_1\,e^{i\sigma})+g\cos\alpha, \tag{7·2·1}$$

and

$$\frac{d^2\zeta}{dt^2}-2i\beta r\frac{d\zeta}{dt}=f\mu_2\,e^{i\sigma}+\frac{d}{dt}(\mu_3\,e^{i\sigma}). \tag{7·2·2}$$

Since r is constant we have, from (3·3·9),

$$\sigma-\sigma_0=r(t-t_0)=\xi, \tag{7·2·3}$$

say.† Equations (7·2·1, 2) can then be written in the forms

$$\frac{d}{dt}(VZ)=f\zeta+g\cos\alpha+f\mu_1\,e^{i(\sigma_0+\xi)} \tag{7·2·4}$$

and

$$\frac{d}{dt}\left\{e^{-2i\beta\xi}\frac{d\zeta}{dt}\right\}=f\mu_2\,e^{i\sigma_0+i(1-2\beta)\xi}+\frac{d}{dt}\{\mu_3\,e^{i(\sigma_0+\xi)}\}\,e^{-2i\beta\xi}. \tag{7·2·5}$$

The quantities $d\zeta/dt$ and ζ can therefore be obtained from (7·2·5) by successive integration, and Z is then given by (7·2·4). We set down the solutions in the succeeding subsections, each disturbing factor being considered separately according to the procedure described at the end of § 4·4. The solutions for a combination of disturbing factors can then be found by addition since the equations are linear. Only the two most important tolerances, namely, α_R and α_C, are considered. The effects of the remaining four can easily be found by similar methods if required.

It will occasionally be possible to simplify the formulae in the following three cases:

(α) ξ small in comparison with unity.
(β) $2\beta\xi$ small, ξ large in comparison with unity.
(γ) $2\beta\xi$ large in comparison with unity.

7·21. *Initial yaw*

Suppose that $Z=0$, $d\zeta/dt=0$ at the instant when the projectile leaves the rails, but that there is an initial yaw

$$\Xi=\Xi_0=\zeta_0.$$

Then, since the right-hand side of (7·2·5) is zero, we have easily

$$Z=\Xi_0\frac{V-V_0}{V}, \tag{7·21·1}$$

$$\Xi=\Xi_0\frac{V_0}{V}, \tag{7·21·2}$$

and

$$d\zeta/dt=0. \tag{7·21·3}$$

7·22. *Initial rate of turn*

Suppose that $Z=\Xi=0$, $d\zeta/dt=\zeta_{01}$ at the instant when the projectile leaves the rails. Then at any later instant, from (7·2·4, 5),

$$Z=\frac{\zeta_{01}f}{4\beta^2r^2V}\{1+2i\beta\xi-e^{2i\beta\xi}\}, \tag{7·22·1}$$

† This angle ξ has no connexion with the angle of the same name which defines the direction of the wind. Owing to the assumptions regarding n^2 the effect of wind is negligible, and so no confusion should arise.

$$\Xi = \frac{\zeta_{01}}{4\beta^2 r^2 V}\{2i\beta r(V_0 - V\,e^{2i\beta\xi}) - f(1-e^{2i\beta\xi})\}, \tag{7·22·2}$$

and

$$d\zeta/dt = \zeta_{01}\,e^{2i\beta\xi}. \tag{7·22·3}$$

In cases (α) and (β) the following more approximate formulae for the deviations and yaw hold:

$$Z = \frac{\zeta_{01}}{2fV}(V-V_0)^2 \tag{7·22·4}$$

and

$$\Xi = \frac{\zeta_{01}}{2fV}(V^2-V_0^2). \tag{7·22·5}$$

7·23. *Gravity*

We assume perfect launch. The effect of gravity tip-off at launch on the later motion can be found from the formulae of §§ 7·21 and 7·22 when Ξ_0, Z_0 and ζ_{01} are known.

From (7·2·4, 5) we have

$$Z = -\Xi = \frac{g(V-V_0)}{fV}\cos\alpha, \tag{7·23·1}$$

and

$$d\zeta/dt = 0. \tag{7·23·2}$$

These formulae show that the projectile's axis remains pointing in the same direction throughout the motion.

7·24. *Malalined thrust*

As in § 6·10, we suppose that the thrust is inclined at a fixed small angle α_R and orientation ϕ_R with respect to the rocket's axis, and assume perfect launch. Then the resulting motion is given by

$$Z = i\alpha_R\,e^{i(\phi_R+\sigma_0)}\frac{mlf^2}{Ar^3V}\left\{-\frac{i\xi}{2\beta}+\frac{1}{1-2\beta}\left(\frac{1}{4\beta^2}e^{2i\beta\xi}-e^{i\xi}\right)-\frac{1+2\beta}{4\beta^2}\right\}, \tag{7·24·1}$$

$$\zeta = i\alpha_R\,e^{i(\phi_R+\sigma_0)}\frac{mlf}{2A\beta(1-2\beta)\,r^2}\{2\beta(e^{i\xi}-1)-(e^{2i\beta\xi}-1)\}, \tag{7·24·2}$$

$$\Xi = \zeta - Z, \tag{7·24·3}$$

and

$$\frac{d\zeta}{dt} = i\alpha_R\,e^{i(\phi_R+\sigma_0)}\frac{mlf}{(1-2\beta)\,Ar}\{e^{i\xi}-e^{2i\beta\xi}\}. \tag{7·24·4}$$

In the cases (α), (β) and (γ) these formulae admit of considerable simplification. Thus

$$\left.\begin{aligned} Z &= -\alpha_R\,e^{i(\phi_R+\sigma_0)}\frac{ml}{Af}\frac{(V-V_0)^3}{6V} && (\alpha)\\ &= -i\alpha_R\,e^{i(\phi_R+\sigma_0)}\frac{ml}{Ar}\frac{(V-V_0)^2}{2V} && (\beta)\\ &= \alpha_R\,e^{i(\phi_R+\sigma_0)}\frac{mlf}{Cr^2}\frac{V-V_0}{V} && (\gamma), \end{aligned}\right\} \tag{7·24·5}$$

$$\left.\begin{aligned} \Xi &= -\alpha_R\,e^{i(\phi_R+\sigma_0)}\frac{ml}{Af}\frac{(V-V_0)^2\,(2V+V_0)}{6V} && (\alpha)\\ &= -i\alpha_R\,e^{i(\phi_R+\sigma_0)}\frac{ml}{Ar}\frac{V^2-V_0^2}{2V} && (\beta)\\ &= \alpha_R\,e^{i(\phi_R+\sigma_0)}\frac{mlf}{Cr^2}\left\{\frac{V_0}{V}-e^{2i\beta\xi}\right\} && (\gamma), \end{aligned}\right\} \tag{7·24·6}$$

and
$$\left.\begin{aligned}\frac{d\zeta}{dt} &= -\alpha_R\,\mathrm{e}^{i(\phi_R+\sigma_0)}\frac{ml}{A}(V-V_0) \quad (\alpha)\\ &= i\alpha_R\,\mathrm{e}^{i(\phi_R+\sigma_0)}\frac{mlf}{Ar}(\mathrm{e}^{i\xi}-1) \quad (\beta).\end{aligned}\right\} \tag{7·24·7}$$

These formulae take into account the moment of the thrust about the centre of gravity, but neglect the effect of the lateral force due to the malalined thrust, i.e. the contribution from μ_1. If it is desired to take this effect, which will usually be of smaller order, into account, $i\alpha_R\,\mathrm{e}^{i(\phi_R+\sigma_0)}f(1-\mathrm{e}^{i\xi})/(\tau V)$ should be added to the expressions for Z, and subtracted from the expressions for Ξ.

7·25. *Displaced longitudinal principal axis of inertia*

In this case we have $\mu_1 = 0$, and

$$\mu_2 = -i\alpha_C\,\mathrm{e}^{i\phi_C}\frac{m(dA/dt)\,(1-2\beta)\,r}{QWA}, \quad \mu_3 = -i\alpha_C\,\mathrm{e}^{i\phi_C}(1-2\beta)\,r.$$

We find the effect of μ_3 on the motion and neglect the contribution from μ_2 since it is of lesser importance in nearly all cases. We have then, from (7·2·4, 5),

$$\mathrm{Z} = -i\alpha_C\,\mathrm{e}^{i(\phi_C+\sigma_0)}\frac{(1-2\beta)f}{rV}\left\{-\frac{i\xi}{2\beta}+\frac{1}{1-2\beta}\left(\frac{1}{4\beta^2}\mathrm{e}^{2i\beta\xi}-\mathrm{e}^{i\xi}\right)-\frac{1+2\beta}{4\beta^2}\right\}, \tag{7·25·1}$$

$$\zeta = -i\alpha_C\,\mathrm{e}^{i(\phi_C+\sigma_0)}\left\{\mathrm{e}^{i\xi}-1-\frac{\mathrm{e}^{2i\beta\xi}-1}{2\beta}\right\}, \tag{7·25·2}$$

and
$$d\zeta/dt = -i\alpha_C\,\mathrm{e}^{i(\phi_C+\sigma_0)}\,r\{\mathrm{e}^{i\xi}-\mathrm{e}^{2i\beta\xi}\}. \tag{7·25·3}$$

Approximate formulae of the same type as those obtained in § 7·24 may be derived, the ratio of the quantities Z, Ξ and $d\zeta/dt$ to their counterparts in that section being

$$-\frac{\alpha_C\,\mathrm{e}^{i\phi_C}}{\alpha_R\,\mathrm{e}^{i\phi_R}}\frac{Ar^2}{mlf}.$$

7·3. Medium or small constant spin and a stabilizing lift moment

In this section we consider the equations of motion for the case of a constant 'medium' or 'small' spin when the lift moment is stabilizing. We use the qualifying adjective 'medium' to denote (*a*) that the spin is not too large, e.g. not great enough to cause overspinning, and (*b*) that it is of a magnitude sufficient to ensure that the number of revolutions made by the projectile during the burning period is large. The spin is called 'small' if (*a*) is satisfied but not (*b*). The condition (*a*) may be expressed more precisely as follows. We require that:

(i) $n^2 > 0$.
(ii) $\beta^2 r^2/n^2 V^2$ is small in comparison with unity between launch and burnt.
(iii) $\beta r f/n^2 V^3$ is small in comparison with unity between launch and burnt.

The condition (*b*) for medium spin is that $\sigma-\sigma_0 = r(t-t_0)$ is large in comparison with unity.

Because of (ii) and (iii) we may put $G(s) = n^2$ in (4·3·8). For, by (4·2·17 to 4·2·19),

$$G(s) = n^2 + \frac{\beta^2 r^2}{V^2} - i\frac{\beta r f}{V^3},$$

since damping and Magnus terms are neglected, and the second and third terms on the right-hand side are small in comparison with the first.

We may therefore solve (4·3·8) exactly as done in § 5·2, replacing p by n wherever it occurs. We then have

$$\mathrm{H} = K_1 \cos n(s-s_0) + K_3 \sin n(s-s_0) + \frac{1}{n}\int_{s_0}^{s} \{nR_2(u) \cos n(s-u) - R_1(u) \sin n(s-u)\}\, du, \quad (7·3·1)$$

where K_1 and K_3 are constants which depend upon the initial conditions and where $R_1(s)$ and $R_2(s)$ are defined by (5·2·2, 3 to 5). The values of K_1 and K_3 can be obtained from (5·2·8, 10). Formulae (5·2·13 to 5·2·15) also hold in this case, and Z and $d\zeta/dt$ may be obtained from them by using (4·3·9) and (4·4·7).

Since r is constant we have

$$P(s) = -i\beta r(t-t_0), \quad (7·3·2)$$

and

$$\Lambda(s) = i\beta r/V. \quad (7·3·3)$$

7·31. *Deviations due to the various disturbing factors*

The angular deviation, yaw and rate of turn at any instant during burning due to the eleven disturbing factors listed at the end of § 4·4 can be found by direct integration as described above, when the spin is medium or small. In the cases (1) to (5), i.e. initial angular deviation, initial yaw, initial rate of turn, gravity and wind, the spin does not enter the formulae except in conjunction with the small parameter† β. Accordingly, in these cases, the angular deviation may be expressed in the form

$$\mathrm{Z} = \mathrm{Z}_0 + i\beta r \mathrm{Z}_1 + (i\beta r)^2 \mathrm{Z}_2 + (i\beta r)^3 \mathrm{Z}_3 + \dots$$
$$\doteq \mathrm{Z}_0 + i\beta r \mathrm{Z}_1 \quad (7·31·1)$$

approximately. Here Z_0, Z_1, Z_2, ... are independent of r and the terms decrease in order of magnitude. Clearly Z_0 is the deviation which is attained when there is no axial spin and can therefore be found immediately from the formulae obtained in § 6 for the case of unrotated motion ($\gamma = 0$).

In the case of small spin $i\beta r\mathrm{Z}_1$, which is in the plane perpendicular to that of Z_0, can usually be neglected. When the spin is medium, however, $i\beta r\mathrm{Z}_1$ though small is not, in general, entirely negligible, and explicit formulae have been derived in terms of the usual Fresnel functions. These formulae are not given here since they are fairly complicated, and since for most purposes an exact knowledge of Z is not required. Where such a knowledge is required it will usually be found to be preferable to obtain the deviation from the more exact formulae of §§ 5 and 8 by step by step methods of integration, the correct damping factors and velocity-time relationship being used.‡

† I.e. the effect of the spin on the motion is due to the precession of the axis and not due directly to the spin about the axis.

‡ This has been done for various 3 in. rockets launched from spiral projectors in order to find the effect of rotation upon (i) wind deviations, (ii) gravity drop and drift, and (iii) drift due to tip-off at launch.

It may be remarked that the *absolute* magnitude of the angular deviation is usually the same as in the unrotated case since $|Z_1|^2\beta^2r^2$ is negligible in comparison with $|Z_0|^2$ in most cases.

Similar conclusions can, of course, be drawn for the yaw Ξ and rate of turn $d\zeta/dt$.

In the case of the six disturbing factors (6) to (11) due to the various tolerances, the above remarks do not apply. For each disturbing factor travels round with the rocket as it rotates about its axis, and the formulae accordingly contain terms in r due directly to the axial spin, in addition to terms in βr attributable to the precessional motion.

A rapidly convergent expansion in terms of r of the type (7·31·1) is not therefore possible unless the spin is sufficiently small. In this case Z can be expressed in a form similar to (7·31·1), namely,

$$Z = Z_0 + irZ_1,$$

where Z_0 is the angular deviation when there is no rotation, and where Z_1 is independent of r and is expressible in terms of Fresnel functions.

Accordingly it is desirable to have approximate formulae for the angular deviation, yaw, and rate of turn due to tolerances, which are valid for medium spin. We obtain such formulae for the two most important tolerances, namely, malalined thrust and displaced longitudinal principal axis of inertia, in the following two subsections.

7·32. *Malalined thrust and medium spin*

We consider here the motion due to the moment of the malalined thrust about the centre of gravity, and assume perfect launch. Accordingly, we take, in the notation of §§ 5·2 and 7·3,

$$R_1(s) = \alpha_R\,e^{i(\phi_R+\sigma_0)}\frac{mlf}{AV}e^{ir'(t-t_0)}, \quad R_2(s) = 0, \tag{7·32·1}$$

where

$$r' = r(1-\beta).$$

Also $K_1 = K_3 = 0$ so that, by (7·3·1),

$$H = -\alpha_R\,e^{i(\phi_R+\sigma_0)}\frac{mlf}{An}\int_{t_0}^{t} e^{ir'(t-t_0)}\sin n(s-u)\,dt_u. \tag{7·32·2}$$

It follows from (4·2·23), (4·4·7) and (5·2·13) that

$$\Xi = \alpha_R\,e^{i(\phi_R+\sigma_0)}\,\Xi_R, \quad d\zeta/dt = \alpha_R\,e^{i(\phi_R+\sigma_0)}\,d\zeta_R/dt, \tag{7·32·3}$$

where

$$\Xi_R = -\frac{mlf}{AnV}e^{i\beta r(t-t_0)}\int_{t_0}^{t} e^{ir'(t_u-t_0)}\sin n(s-u)\,dt_u, \tag{7·32·4}$$

and

$$\frac{d\zeta_R}{dt} = -\frac{mlf}{A}\,e^{i\beta r(t-t_0)}\int_{t_0}^{t} e^{ir'(t_u-t_0)}\left\{\cos n(s-u) + \frac{i\beta r}{nV}\sin n(s-u)\right\}dt_u. \tag{7·32·5}$$

Also, by (4·3·9), (5·2·13) and (7·32·4),

$$Z = \alpha_R\,e^{i(\phi_R+\sigma_0)}\,Z_R = \alpha_R\,e^{i(\phi_R+\sigma_0)}\frac{mlf}{A}\Phi(t_0, t), \tag{7·32·6}$$

where

$$e^{irt_0}\Phi(t_0,t) = \frac{1}{nV}\int_{t_0}^{t} e^{i\beta rt+ir't_u}\sin n(s-u)\,dt_u - \int_{t_0}^{t} dt_u\int_{t_0}^{t_u} e^{i\beta rt_u+ir't_v}\cos n(u-v)\,dt_v - \frac{i\beta r}{nf}\int_{t_0}^{t} dt_u\int_{t_0}^{t_u} e^{i\beta rt_u+ir't_v}\sin n(u-v)\,\frac{dt_v}{t_v}. \tag{7·32·7}$$

The integrals in (7·32·7) may be evaluated approximately by integration by parts and by using the second mean-value theorem to estimate the new integrals obtained. Thus, for example, for the third integral on the right of (7·32·7) we use the approximation

$$\int_{t_0}^{t} e^{ir't_u} \sin n(s-u) \frac{dt_u}{t_u} \doteqdot e(r't_0) \sin n(s-s_0),$$

where $e(u)$ is defined by (6·82·2); the corresponding approximations for the first two integrals are somewhat simpler and we omit the details. We obtain, finally,

$$\Phi(t_0, t) = -\frac{i}{r'} \sqrt{\left(\frac{\pi}{nf}\right)} e^{i\beta r(t-t_0)} \left\{ G(v_0, v) + i\frac{\beta r}{nV} A(v_0) \right\} + j(t_0, t), \tag{7·32·8}$$

where $j(t_0, t)$ is an error term. Hence, by condition (*a*) (ii),

$$|Z| = \alpha_R |Z_R| = \alpha_R \frac{ml}{Ar} \sqrt{\left(\frac{\pi f}{n}\right)} G(v_0, v), \tag{7·32·9}$$

approximately, provided that $|j(t_0, t)|$ is small in comparison with

$$\sqrt{\left(\frac{\pi}{nf}\right)} \frac{G(v_0, v)}{r}.$$

In view of the assumptions made in § 7·3 and the approximations used above, the range of application of the formula (7·32·9) is not very wide, and even in favourable circumstances the error may be as great as 20 %. For the purpose of estimating dispersion, however, this inaccuracy is unimportant since the angle α_R is rarely known to within even this degree of certainty.

We have, similarly,

$$\Xi_R = -i \frac{mlf}{nVrA} e^{i\beta r(t-t_0)} \sin n(s-s_0), \tag{7·32·10}$$

and

$$\frac{d\zeta_R}{dt} = -i\frac{mlf}{Ar} e^{i\beta r(t-t_0)} \left\{ \cos n(s-s_0) - e^{ir'(t-t_0)} + \frac{i\beta r}{nV_0} \sin n(s-s_0) \right\}, \tag{7·32·11}$$

approximately.

7·33. *Displacement of the longitudinal principal axis of inertia*

We assume perfect launch and consider the motion due to the displaced inertial axis. This consists of two parts; we consider only the contribution from the tolerance function μ_3 and neglect the less important contribution from μ_2. We then have, in the notation of §§ 5·2 and 7·3,

$$R_1(s) = -\alpha_C e^{i(\phi_C+\sigma_0)} (1-2\beta) \frac{\beta r^2}{V} e^{ir'(t-t_0)}, \tag{7·33·1}$$

$$R_2(s) = -i\alpha_C e^{i(\phi_C+\sigma_0)} (1-2\beta) r e^{ir'(t-t_0)}, \tag{7·33·2}$$

and

$$K_1 = 0, \quad K_3 = i\alpha_C e^{i(\phi+\sigma_0)} (1-2\beta) \frac{r}{n}. \tag{7·33·3}$$

It follows from (7·3·1), (4·4·7) and (4·3·9) that

$$\Xi = \alpha_C e^{i(\phi_C+\sigma_0)} \Xi_C, \quad d\zeta/dt = \alpha_C e^{i(\phi_C+\sigma_0)} d\zeta_C/dt, \tag{7·33·4}$$

and

$$Z = \alpha_C e^{i(\phi_C+\sigma_0)} Z_C, \tag{7·33·5}$$

where

$$\Xi_C = -i e^{-i\sigma_0} (1-2\beta) r x_3(1), \tag{7·33·6}$$

$$d\zeta_C/dt = -i e^{-i\sigma_0} (1-2\beta) r y_3(1), \tag{7·33·7}$$

and

$$Z_C = -i e^{-i\sigma_0} (1-2\beta) r z_3(1), \tag{7·33·8}$$

the functions x_3, y_3, z_3 being defined by (5·7·8, 11, 15).

We evaluate here only the function $z_3(1)$. The functions $x_3(1)$ and $y_3(1)$ may be found by similar methods and yield simpler results as they involve no repeated integrals. As in § 7·32 we assume that βrt is small and that rt is large. Then we have, from (5·7·15),

$$e^{-i\sigma_0} z_3(1) = \frac{\sin n(s-s_0)}{nV} - \int_{s_0}^{s} \cos n(u-s_0) \frac{du}{V} + \int_{s_0}^{s} e^{ir(t_u-t_0)} \frac{du}{V}$$
$$-\frac{1}{V}\int_{s_0}^{s} e^{ir(t_u-t_0)} \cos n(s-u)\, du - n\int_{s_0}^{s} \frac{du}{V_u} \int_{s_0}^{u} e^{ir(t_v-t_0)} \sin n(u-v)\, dv,$$

approximately. The first two terms on the right-hand side predominate owing to the absence of any e^{irt} term and give

$$z_3(1) \doteqdot \sqrt{\left(\frac{\pi}{nf}\right)}\, G(v_0, v). \tag{7·33·9}$$

Hence, we have, from (7·33·5, 8, 9),

$$Z = \alpha_C e^{i(\phi_C+\sigma_0)} Z_C$$
$$= -i\alpha_C e^{i(\phi_C+\sigma_0)} r \sqrt{\left(\frac{\pi}{nf}\right)}\, G(v_0, v), \tag{7·33·10}$$

approximately.

Since, by (7·32·9) and (7·33·10),

$$\left|\frac{Z_C}{Z_R}\right| \doteqdot r^2 \Big/ \frac{ml}{A} f, \tag{7·33·11}$$

it follows that, for high spins, a displaced axis of inertia has a greater effect upon the deviation than a malalined thrust.†

7·4. Variable spin

In §§ 7·2 and 7·3 we have obtained various solutions of the equations of motion under the assumption of constant spin. It is possible, however, to use the same methods for other forms of spin. Thus, for example, if the spin is 'medium' or 'small' (see beginning of § 7·3) we may put $G(s) = n^2$ and use the formulae of § 5·2 together with (4·2·23), (4·4·7) and (4·3·9) to find the yaw, rate of turn and angular deviation at any instant (with $p = n$). The correct expressions for $P(s)$, $\Lambda(s)$ and σ must, of course, be substituted.

For example, if the spin is proportional to Vs, we have‡

$$r = \gamma^* t^3$$

say, if f is constant throughout burning, and then

$$\sigma = \sigma_0 + \tfrac{1}{4}\gamma^*(t^4 - t_0^4), \quad P(s) = -i\beta(\sigma-\sigma_0) \quad \text{and} \quad \Lambda(s) = i\beta\gamma^* t^2/f.$$

In the case of the disturbing factors (1) to (5) listed at the end of § 4·4, formulae of the same type as (7·31·1) can be obtained for the angular deviation, etc., by expanding $e^{-i\beta\sigma}$ in powers of $\beta\gamma^*$. The same process of successive integration by parts as that used in § 7·32 may also be employed to obtain asymptotic formulae for the deviations due to tolerances when the spin is medium.

Since there has been, up to the present time, no practical requirement for formulae of this type, it has not been considered necessary to include them here.

† For the 3 in. rocket with tubular charge and medium spin ($r \doteqdot 90$ rad./sec.) the ratio on the right of (7·33·11) is approximately 6.

‡ When the spin is imparted by offset fins alone, r is of this form in the early stages of flight (see § 9·5).

8. THE GENERAL SOLUTION OF THE EQUATIONS AND THE STABILITY OF THE MOTION

8·1. Introduction

In §§ 5 to 7 we have investigated the motion of a rocket under various restrictive assumptions (Assumptions A, B and C of § 3·6). The most important of these assumptions were those concerning the forms of the spin, the acceleration and the aerodynamic coefficients, and they were made (i) because they were believed to hold with reasonable accuracy for many existing types of rocket, and (ii) in order that the equations of motion might be solved in a form suitable for practical use.

It is possible, however, to obtain general solutions of the equations without making any of these special restrictive assumptions.† The solutions are necessarily of an approximate form, but in any practical case the errors involved are likely to be negligible in comparison with the errors arising from an inadequate knowledge of the necessary numerical data (e.g. the magnitude of the aerodynamic Magnus and damping coefficients).‡ Although the general solutions which we shall obtain in § 8·2 are not of a type suitable for quick or routine calculations—partly because of their complicated form, but mainly because of the numerous parameters whose numerical values will rarely be known to any degree of accuracy—they are of very great value in determining the general character of the motion. In particular, the conditions which must be satisfied if stable motion is to ensue can be deduced from them. As will be seen in § 8·5, these conditions are of a more complicated form than is generally realized by rocket designers, since the aerodynamic lift moment is by no means the only important factor involved.

8·2. The general solution

As stated in the previous section, we suppose that the fundamental Assumptions A hold, but we do not make the more restrictive Assumptions B and C (see § 3·6). Our starting point is therefore § 4, and, in particular, equations (4·3·8) and (4·3·9), which determine respectively the motion of the axis about the tangent to the trajectory and the angular deviation from the line of departure. Since the angular deviation Z can be found from the second equation when $\mathrm{H}(s)$ and $\mathrm{H}'(s)$ are known, by simple integration, we need only solve the first equation for $\mathrm{H}(s)$.

The equation (4·3·8) may be written, in the notation of (5·2·1), as

$$\frac{d^2\mathrm{H}}{ds^2}+G(s)\,\mathrm{H}=R(s). \tag{8·2·1}$$

Thus the behaviour of H (and therefore of the yaw) is dependent upon the form of the function $G(s)$ defined by (4·2·17 to 4·2·19). We shall obtain an approximate solution of (8·2·1) which is valid when $G(s)$ is a slowly varying function of s which does not vanish during the part of the motion under consideration. This is likely to be the case in the great majority of the cases considered, provided that the rocket is properly designed; we indicate, however, in § 8·5, how the argument may be modified in cases where this assumption does not hold. A more precise statement of our assumptions regarding $G(s)$ is now given.

† I.e. without making Assumptions B and C. The Assumptions A are fundamental to the theory.

‡ This refers to ground-fired rockets only. For finless rockets fired from aircraft finer approximations may be necessary. See § 8·5.

We assume that the function $G(s)$ has the following properties in the interval† $s_0 \leqslant s \leqslant s_1$.

(*a*) $G(s)$ is defined uniquely and $G'(s)$ and $G''(s)$ both exist.

(*b*) $G(s) \neq 0$ at every point.

(*c*) Either $-\pi < \arg G(s) \leqslant \pi$, or $-\pi \leqslant \arg G(s) < \pi$.

(*d*) The function

$$G^*(s) = \frac{1}{4}\frac{G''(s)}{G(s)} - \frac{5}{16}\left\{\frac{G'(s)}{G(s)}\right\}^2 \tag{8·2·2}$$

is small in comparison with $G(s)$.

The reason for the last somewhat complicated assumption is that the solutions which we shall obtain will satisfy the modified equation

$$\frac{d^2\mathrm{H}}{ds^2} + \{G(s) + G^*(s)\}\,\mathrm{H} = R(s), \tag{8·2·3}$$

and will therefore be approximate solutions of (8·2·1) provided that $G^*(s)$ can be neglected in comparison with $G(s)$.

Put
$$G(s) = G, \quad G(s_0) = G_0, \quad G'(s_0) = G_{01},$$

and write
$$g(u, v) = \int_u^v [G(w)]^{\frac{1}{2}}\,dw \tag{8·2·4}$$

for any u, v satisfying $s_0 \leqslant u \leqslant v \leqslant s_1$. Because of the property (*c*) above, the function $[G(w)]^{\frac{1}{2}}$ is defined uniquely, and therefore, since it is integrable by (*a*), $g(u, v)$ exists as a uniquely defined function of u and v.

The procedure which we adopt to solve (8·2·1) is based upon the Jeffreys phase-integral solution (see Jeffreys (1925) and also Kelley, McShane & Reno (1949?)). We first investigate the homogeneous equation

$$\frac{d^2\mathrm{H}}{ds^2} + G(s)\,\mathrm{H} = 0. \tag{8·2·5}$$

Put
$$\mathrm{H}(s) = \exp\left\{i\int_{s_0}^{s} \eta(u)\,du\right\}. \tag{8·2·6}$$

Then we have
$$\mathrm{H}' = i\eta\mathrm{H}, \quad \mathrm{H}'' = (i\eta' - \eta^2)\,\mathrm{H},$$

and therefore, by (8·2·5),
$$i\eta' - \eta^2 + G = 0.$$

Hence, if we put $\eta = G^{\frac{1}{2}} + \epsilon$ and assume that ϵ^2 and ϵ' can be neglected, we obtain

$$\tfrac{1}{2}iG^{-\frac{1}{2}}G' - 2\epsilon G^{\frac{1}{2}} = 0,$$

so that
$$\epsilon = \tfrac{1}{4}iG'G^{-1},$$
approximately. Therefore, by (8·2·6),

$$\mathrm{H} = G^{-\frac{1}{4}}\,e^{ig(s_0,\, s)}$$

is an approximate solution of (8·2·5). In an exactly similar way we obtain the second approximate solution

$$\mathrm{H} = G^{-\frac{1}{4}}\,e^{-ig(s_0,\, s)}.$$

These two solutions are clearly linearly independent. Accordingly, we may take as the general solution of (8·2·5)

$$\mathrm{H} = \left\{\frac{G_0}{G}\right\}^{\frac{1}{4}}\{K_1 \cos g(s_0, s) + K_2 \sin g(s_0, s)\}. \tag{8·2·7}$$

† The point s_1 will usually be the position of burnt, but is used here for the end-point of the arc of the trajectory which it is desired to investigate.

This solution is, of course, only approximate; it can be verified by differentiation that it is in fact the general solution of the modified equation

$$\frac{d^2\mathrm{H}}{ds^2}+\{G(s)+G^*(s)\}\,\mathrm{H}=0.$$

The function $[G_0/G]^{\frac{1}{4}}$ and all other fourth roots appearing later on are uniquely defined because of the property (c) of $G(s)$.

In order to obtain the general solution of (8·2·1) from the solution of (8·2·5) we must find the Wronskian of the two functions

$$\mathrm{H}^{(i)}=\left(\frac{G_0}{G}\right)^{\frac{1}{4}}\cos g(s_0,s),\quad \mathrm{H}^{(ii)}=\left(\frac{G_0}{G}\right)^{\frac{1}{4}}\sin g(s_0,s).$$

It is

$$\mathrm{H}^{(i)}\mathrm{H}^{(ii)\prime}-\mathrm{H}^{(i)\prime}\mathrm{H}^{(ii)}=G_0^{\frac{1}{2}}.$$

Accordingly, the general solution of the equation (8·2·1) is

$$\mathrm{H}(s)=\left(\frac{G_0}{G}\right)^{\frac{1}{4}}\{K_1\cos g(s_0,s)+K_2\sin g(s_0,s)\}+\int_{s_0}^{s}\frac{R(u)}{[G(s)\,G(u)]^{\frac{1}{4}}}\sin g(u,s)\,du, \tag{8·2·8}$$

approximately. Here K_1 and K_2 are arbitrary constants which depend upon the initial conditions. By (4·4·8, 9) we have

$$K_1=\mathrm{H}_0=V_0\Xi_0+w_1(s_0), \tag{8·2·9}$$

and

$$\begin{aligned}K_2&=\tfrac{1}{4}G_{01}G_0^{-\frac{3}{4}}\mathrm{H}_0+G_0^{-\frac{1}{2}}\mathrm{H}_{01}\\&=\tfrac{1}{4}G_{01}G_0^{-\frac{3}{4}}\{V_0\Xi_0+w_1(s_0)\}+G_0^{-\frac{1}{2}}\Big\{\zeta_{01}-\Lambda_0V_0\Xi_0-\Lambda_0w_1(s_0)\\&\qquad-\frac{1}{V_0}[g\cos\alpha+f_0\mu_1(s_0)\,e^{i\sigma_0}-V_0w_1'(s_0)]\Big\}.\end{aligned} \tag{8·2·10}$$

It follows from (8·2·8) that

$$\begin{aligned}\mathrm{H}'(s)=\left(\frac{G_0}{G}\right)^{\frac{1}{4}}&[\{K_2G^{\frac{1}{2}}-\tfrac{1}{4}K_1G'G^{-1}\}\cos g(s_0,s)-\{K_1G^{\frac{1}{2}}+\tfrac{1}{4}K_2G'G^{-1}\}\sin g(s_0,s)]\\&+\int_{s_0}^{s}\frac{R(u)}{[G(s)\,G(u)]^{\frac{1}{4}}}\left\{G^{\frac{1}{2}}(s)\cos g(u,s)-\frac{G'(s)}{4G(s)}\sin g(u,s)\right\}du.\end{aligned} \tag{8·2·11}$$

Because of our assumption regarding $G^*(s)$ the terms involving $G'(s)$ in (8·2·10, 11) will be of smaller order than the remaining terms. If we make the additional assumption regarding $G(s)$ that

(d') the function $\tfrac{1}{4}G'(s)\,[G(s)]^{-\frac{3}{2}}$

is small in comparison with unity,† equation (8·2·11) becomes

$$\mathrm{H}'(s)=(G_0G)^{\frac{1}{4}}[K_2\cos g(s_0,s)-K_1\sin g(s_0,s)]+\int_{s_0}^{s}\left[\frac{G(s)}{G(u)}\right]^{\frac{1}{4}}R(u)\cos g(u,s)\,du, \tag{8·2·12}$$

and we may drop the first term on the right of (8·2·10). Also equations (8·2·8) and (8·2·12) may then be put into the slightly more convenient forms

$$\begin{aligned}\mathrm{H}(s)=&\left(\frac{G_0}{G}\right)^{\frac{1}{4}}\{K_1\cos g(s_0,s)+K_3\sin g(s_0,s)\}\\&+\int_{s_0}^{s}[G(u)\,G(s)]^{-\frac{1}{4}}\{G^{\frac{1}{2}}(u)\,R_2(u)\cos g(u,s)-R_1(u)\sin g(u,s)\}\,du,\end{aligned} \tag{8·2·13}$$

† The property (d') is not always a consequence of the property (d) since the fact that $[G'G^{-\frac{3}{2}}]^2$ is sufficiently small and can be neglected does not necessarily imply that $G'G^{-\frac{3}{2}}$ can also be neglected.

and $$H'(s) = (G_0 G)^{\frac{1}{4}} \{K_3 \cos g(s_0, s) - K_1 \sin g(s_0, s)\} + R_2(s)$$
$$- \int_{s_0}^{s} \left[\frac{G(s)}{G(u)}\right]^{\frac{1}{4}} \{G^{\frac{1}{2}}(u)\, R_2(u) \sin g(u, s) + R_1(u) \cos g(u, s)\}\, du, \quad (8{\cdot}2{\cdot}14)$$

in terms of the functions $R_1(s)$ and $R_2(s)$ which are defined in (5·2·3 to 5·2·5). Here K_1 is given by (8·2·9), and
$$K_3 = K_2 - G_0^{-\frac{1}{4}} R_2(s_0)$$
$$= G_0^{-\frac{1}{4}} \{\zeta_{01} - \Lambda_0 V_0 \Xi_0 - \Lambda_0 w_1(s_0) - \mu_3(s_0)\, e^{i\sigma_0}\}. \quad (8{\cdot}2{\cdot}15)$$

It will be observed that the solutions obtained for H and H′ agree exactly with those obtained in § 5, where it was assumed that $G(s) = p^2 =$ constant.

8·3. The form of the phase integral

Since the function $g(s_0, s)$ will, in general, be complex, $\cos g(s_0, s)$ and $\sin g(s_0, s)$ will not always be real. Write
$$g(u, v) = g_1(u, v) - i g_2(u, v), \quad (8{\cdot}3{\cdot}1)$$
where $$g_1(u, v) = \Re g(u, v), \quad g_2(u, v) = -\Im g(u, v).$$
Then
$$\left.\begin{aligned} \cos g &= \cos g_1 \cosh g_2 + i \sin g_1 \sinh g_2, \\ \sin g &= \sin g_1 \cosh g_2 - i \cos g_1 \sinh g_2. \end{aligned}\right\} \quad (8{\cdot}3{\cdot}2)$$
and

Put also $$G^{\frac{1}{2}}(s) = g_1^*(s) - i g_2^*(s). \quad (8{\cdot}3{\cdot}3)$$
Then we have, by (4·2·17),
$$G = \frac{1}{V^2}(G_1 - i r G_2) = (g_1^* - i g_2^*)^2,$$
so that $$g_1^{*2} - g_2^{*2} = \frac{1}{V^2} G_1, \quad 2 g_1^* g_2^* = \frac{r G_2}{V^2}. \quad (8{\cdot}3{\cdot}4)$$

Hence, if $$G_3(s) = V^2 \mid G(s) \mid = \{G_1^2(s) + r^2 G_2^2(s)\}^{\frac{1}{2}}, \quad (8{\cdot}3{\cdot}5)$$
then $G_3 \geqslant \mid G_1 \mid$ and we have, from (8·3·4) and the property (*c*) of $G(s)$,
$$g_1^*(s) = \frac{1}{V}\{\tfrac{1}{2} G_3(s) + \tfrac{1}{2} G_1(s)\}^{\frac{1}{2}}, \quad g_2^*(s) = \pm \frac{1}{V}\{\tfrac{1}{2} G_3(s) - \tfrac{1}{2} G_1(s)\}^{\frac{1}{2}}, \quad (8{\cdot}3{\cdot}6)$$

where the ambiguous sign is to be taken positive when $rG_2 \geqslant 0$ and negative when $rG_2 < 0$. It is to be observed that $g_1^*(s)$ is always positive or zero. It follows, from (8·3·3) and (8·3·6), that
$$g_1(u, v) = \int_u^v [\tfrac{1}{2}\{G_3(w) + G_1(w)\}]^{\frac{1}{2}} \frac{dw}{V_w}, \quad (8{\cdot}3{\cdot}7)$$

and $$g_2(u, v) = \pm \int_u^v [\tfrac{1}{2}\{G_3(w) - G_1(w)\}]^{\frac{1}{2}} \frac{dw}{V_w}. \quad (8{\cdot}3{\cdot}8)$$

In nearly all cases occurring in practice of rockets fired from the ground $G(s)$ satisfies the two additional conditions:

(*e*) $G_1(s) > 0$,

and

(*f*) $r^2 G_2^2(s)/G_1^2(s)$ small in comparison with unity.

When these conditions hold, the formulae admit of considerable simplification. For we then have

$$G_3 = G_1 + \frac{r^2 G_2^2}{2G_1},$$

approximately, so that $$g_1^*(s) = \frac{1}{V}\{G_1(s)\}^{\frac{1}{2}}, \quad g_2^*(s) = \frac{rG_2(s)}{2V\{G_1(s)\}^{\frac{1}{2}}}. \tag{8·3·9}$$

It follows that $$g_1(u,v) = \int_u^v \{G_1(w)\}^{\frac{1}{2}} \frac{dw}{V_w}, \tag{8·3·10}$$

and $$g_2(u,v) = \frac{1}{2}\int_u^v \frac{rG_2(w)}{V_w\{G_1(w)\}^{\frac{1}{2}}}\, dw. \tag{8·3·11}$$

We shall not, however, use these approximate relations.

8·4. Stability

The object of this section is to consider the stability of rockets. We are chiefly interested in the stability during burning, since the motion after burnt is no different from that of a shell, and the theory of the external ballistics of shells may therefore be used to investigate the stability in this part of the trajectory.†

Before it is possible to obtain precise criteria for stability it is necessary to state in definite terms what we propose to mean by stable motion. Since stability is a matter of degree and depends upon a variety of factors, such as the intended purpose of the projectile, it is not obvious what is the best definition to adopt. Although in exceptional cases projectiles which develop large yaws may reach their objectives as intended, it is clear that stability conditions must be fairly restrictive if they are to be of use in designing rockets and in predicting performance. Accordingly, we shall say that the motion of a rocket is stable if the yaw and rate of yawing remain small during the part of the trajectory under consideration.

8·41. *The conditions for stability in their most general form*

The yaw is given by $$\Xi = \mathrm{H}\frac{e^{-P(s)}}{V} - \frac{w_1}{V}. \tag{8·41·1}$$

We shall suppose that $G(s)$ possesses properties (a) to (d') of § 8·2. Then $\mathrm{H}(s)$ and $\mathrm{H}'(s)$ are given by (8·2·8, 12); since these equations are of a similar form it suffices to consider $\mathrm{H}(s)$ when investigating stability.

Write $$\mathrm{H}(s) = \mathrm{H}^{(1)}(s) + \mathrm{H}^{(2)}(s), \tag{8·41·2}$$

where $\mathrm{H}^{(1)}(s)$ and $\mathrm{H}^{(2)}(s)$ are respectively the first and second expressions on the right of (8·2·13). Also, put

$$P_1(s) = \Re P(s) = \int_{s_0}^s \left(\alpha_2 + \frac{\kappa}{V_u}\right) du. \tag{8·41·3}$$

We consider the contributions of $\mathrm{H}^{(1)}$ and $\mathrm{H}^{(2)}$ to the yaw separately, and write

$$\Xi = \Xi^{(1)} + \Xi^{(2)} - \frac{w_1}{V}, \tag{8·41·4}$$

† See, for example, Kelley, McShane & Reno (1949?, chapter 12). With the exception of a few extra factors, such as acceleration damping and jet damping, the conditions for stability after burnt differ very little from those which apply during burning, and the analysis of § 8·2 may be used with only minor modifications. The chief differences which appear arise from the fact that, due to gravity, Z does not remain small throughout the motion, and consequently sin Θ is no longer small.

where $$\Xi^{(1)} = \mathrm{H}^{(1)}\,\mathrm{e}^{-P(s)}/V, \quad \Xi^{(2)} = \mathrm{H}^{(2)}\,\mathrm{e}^{-P(s)}/V. \tag{8·41·5}$$

Since w_1/V is small by Assumption A 1, we need only examine the forms of $\Xi^{(1)}$ and $\Xi^{(2)}$.

If we consider the amplitude of $\mathrm{H}^{(1)}$ it is clear that a necessary and sufficient condition for $\Xi^{(1)}$ to remain small is that

$$\frac{1}{V}\left|\frac{G_0}{G}\right|^{\frac{1}{4}} \mathrm{e}^{|g_2(s_0,\,s)|-P_1(s)}$$

must be either a decreasing function of s or else must increase very slowly. Since, by our assumptions, $G(s)$ is a slowly varying function of s, we may omit the factor $|G_0/G|^{\frac{1}{4}}$ in the above expression. On differentiating with respect to s we deduce, from (8·3·8) and (8·41·3), that

$$[\tfrac{1}{2}\{G_3(s)-G_1(s)\}]^{\frac{1}{2}} \leqslant G_4(s)+\epsilon \tag{8·41·6}$$

where ϵ is a small non-negative quantity which is related to the maximum permissible rate of increase of yaw with the time,† and

$$G_4(s) = \alpha_2 V + \kappa + \frac{f}{V}. \tag{8·41·7}$$

By (8·3·5), this is equivalent to

Condition A: $$S = 4(G_4+\epsilon)^2\{G_1+(G_4+\epsilon)^2\} - r^2G_2^2 \geqslant 0. \tag{8·41·8}$$

This is our first stability condition in its most general form.

We now consider the contribution of $\mathrm{H}^{(2)}(s)$ to the yaw, i.e. we have to examine the behaviour of the function

$$\Xi^{(2)} = \frac{1}{V}\,\mathrm{e}^{-P(s)}\int_{s_0}^{s}[G(u)\,G(s)]^{-\frac{1}{4}}\{G^{\frac{1}{2}}(u)\,R_2(u)\cos g(u,s) - R_1(u)\sin g(u,s)\}\,du \tag{8·41·9}$$

during burning. It is not possible to deduce the conditions which must be satisfied if this is to be small in such a precise form as the Condition A just obtained, since the behaviour of the integral depends upon so many different factors. By (5·2·1), the function $R(s)$—and therefore each of $R_1(s)$ and $R_2(s)$—can be expressed in the form

$$R(s) = R_{\mathrm{I}}(s) - R_{\mathrm{II}}(s), \tag{8·41·10}$$

where $R_{\mathrm{I}}(s)$ contains the contributions of wind and gravity, and $R_{\mathrm{II}}(s)$ those of tolerances. We consider those two parts separately and write

$$\Xi^{(2)} = \Xi^{(2)}_{\mathrm{I}} + \Xi^{(2)}_{\mathrm{II}}.$$

Then, since $$\frac{V_u}{V} = \exp\left\{-\int_u^s \frac{f\,dv}{V_v^2}\right\},$$

$\Xi^{(2)}_{\mathrm{I}}$ may be written in the form

$$\Xi^{(2)}_{\mathrm{I}} = \int_{s_0}^{s} k_1(u,s)\exp\{-q_{11}(u,s)+iq_{12}(u,s)\}\,du - \int_{s_0}^{s} k_2(u,s)\exp\{-q_{21}(u,s)+iq_{22}(u,s)\}\,du, \tag{8·41·11}$$

† If ϵ' is this maximum rate, we may take ϵ so that $\epsilon\,\mathrm{e}^{\epsilon(t_1-t_0)} = \epsilon'$. In applications it will usually be convenient to choose $\epsilon' = \epsilon = 0$.

where

$$\left.\begin{aligned} q_{11}(u,s) &= \int_u^s \left\{\frac{G_4(v)}{V_v} + g_2^*(v)\right\} dv, \\ q_{21}(u,s) &= \int_u^s \left\{\frac{G_4(v)}{V_v} - g_2^*(v)\right\} dv, \\ q_{12}(u,s) &= \int_u^s \left\{\beta_2 \frac{r}{V_v} - g_1^*(v)\right\} dv, \\ q_{22}(u,s) &= \int_u^s \left\{\beta_2 \frac{r}{V_v} + g_1^*(v)\right\} dv, \end{aligned}\right\} \qquad (8\cdot41\cdot12)$$

and where both the real and the imaginary parts of $k_1(u,s)$ and $k_2(u,s)$ are monotonic or slowly varying functions of u and of s. Also $k_1(u,s)$ and $k_2(u,s)$ are each of the order of magnitude of $g\cos\alpha/V_u^2$ by (5·2·4, 5) and (8·41·9).

If $\Xi_{\mathrm{I}}^{(2)}$ is to be small, it is necessary (i) that the amplitudes of the oscillating terms in (8·41·11) should be decreasing or only very slowly increasing functions of s and (ii) that the frequency of each of the two oscillatory terms

$$e^{iq_{12}} \quad \text{and} \quad e^{iq_{22}}$$

should not be too small. The first of these two conditions is exactly the same as that obtained for $\Xi^{(1)}$, namely, Condition A above. The second condition is of a different type. It is necessary in order that the integrals may not build up to a large value as the arc length increases. For this will occur if each of the two *rates of precession*

$$\left|\frac{d}{dt}q_{12}\right| \quad \text{and} \quad \left|\frac{d}{dt}q_{22}\right|$$

is small. In order to show this it suffices to consider the first term on the right of (8·41·11). It can be written in the form

$$\int_{s_0}^s k_1(u,s)\, e^{-q_{11}(u,s)} \frac{d\, e^{iq_{12}(u,s)}}{i\left\{\beta_2 \dfrac{r}{V_u} - g_1^*(u)\right\}}.$$

By the second mean value theorem, this is of the order of magnitude of

$$\max_{s_0 \leqslant u \leqslant s} \frac{g\cos\alpha}{V_u \,|\, \beta_2 r - V_u g_1^*(u) \,|}$$

when Condition A is satisfied, since $k_1(u,s)$ and $e^{-q_{11}(u,s)}$ are monotonic or slowly varying functions of u. The second term on the right of (8·41·11) is, by a similar argument, of the order of magnitude of

$$\max_{s_0 \leqslant u \leqslant s} \frac{g\cos\alpha}{V_u \,|\, \beta_2 r + V_u g_1^*(u) \,|}.$$

These estimates are, of course, meaningless if $\beta_2 r \pm V_u g_1^*(u)$ vanishes in the range considered.

On the other hand, we have, from (8·41·11), by taking the absolute value of the integrand, the crude result that $\Xi_{\mathrm{I}}^{(2)}$ is of the order of magnitude of

$$\int_{s_0}^s \frac{g\cos\alpha}{V_u^2}\, du. \qquad (8\cdot41\cdot13)$$

Hence the second stability condition may be stated as follows:

Condition B: If the angle (8·41·13) *is not small, then*

$$\big|\, |\beta_2 r| - V g_1^*(s) \,\big|$$

must be large in comparison with $g\cos\alpha/V$.

This condition is actually more stringent than may be absolutely necessary to ensure that $\Xi_{\mathrm{I}}^{(2)}$ is small. Thus, when (8·41·13) is not small it may happen that $\Xi_{\mathrm{I}}^{(2)}$ remains small even when one of $\beta_2 r \pm V g_{\mathrm{I}}^*(s)$ vanishes at some point of the interval. This, however, can only happen, in general, if

$$\frac{d}{ds}\left\{\beta_2 \frac{r}{V} \pm g_{\mathrm{I}}^*(s)\right\}$$

remains sufficiently large throughout the interval. We rule out this possibility as being unlikely to occur in practice.

We have now to consider the contribution to the yaw from the part $R_{\mathrm{II}}(s)$, i.e. from the tolerances. Since each term in $R_{\mathrm{II}}(s)$ contains $e^{i\sigma}$ as a factor, we write $\Xi_{\mathrm{II}}^{(2)}$ in the form

$$\Xi_{\mathrm{II}}^{(2)} = \int_{s_0}^{s} K_1(u,s) \exp\{-Q_{11}(u,s) + iQ_{12}(u,s)\}\, du - \int_{s_0}^{s} K_2(u,s) \exp\{-Q_{21}(u,s) + iQ_{22}(u,s)\}\, du, \qquad (8\cdot41\cdot14)$$

where

$$\left.\begin{aligned} Q_{11}(u,s) &= q_{11}(u,s), \quad Q_{21}(u,s) = q_{21}(u,s), \\ Q_{12}(u,s) &= \int_u^s \left\{(1-\beta_2)\frac{r}{V_v} + g_{\mathrm{I}}^*(v)\right\} dv, \\ Q_{22}(u,s) &= \int_u^s \left\{(1-\beta_2)\frac{r}{V_v} - g_{\mathrm{I}}^*(v)\right\} dv. \end{aligned}\right\} \qquad (8\cdot41\cdot15)$$

We now make the assumption that none of the twelve tolerance angles α_P, ϕ_P oscillates rapidly during the motion ($P = C, G, L, M, N, R$). Then $K_1(u,s)$ and $K_2(u,s)$ are monotonic or slowly varying functions of u and s of small absolute magnitude. Write ω_T for the maximum value of $V_u\,|\,K_1(u,s)\,|$ and $V_u\,|\,K_2(u,s)\,|$ during the range considered. Then ω_T depends upon the spin, velocity, acceleration and the six tolerances α_P. It has the dimensions of an angular velocity, and we may regard it as measuring the total amount of asymmetry possessed by the projectile. Then, exactly as for $\Xi_{\mathrm{I}}^{(2)}$, we require that Condition A shall hold, and obtain the third stability condition, namely

Condition C: If

$$\int_{t_0}^{t} \omega_T\, dt_u$$

is not small, then

$$|\,|\,(1-\beta_2)\,r\,| - Vg_{\mathrm{I}}^*(s)\,|$$

must be large in comparison with ω_T.

The condition is again slightly more stringent than may be necessary.

It is clear that if the assumptions made regarding the slow variation of $\alpha_P\, e^{i\phi_P}$ do not hold, and if, for example, ν_P is the frequency of oscillation of one of its modes of greatest amplitude, then the second part of Condition C will be replaced by the condition that

$$|\,|\,(1-\beta_2)\,r + \nu_P\,| - Vg_{\mathrm{I}}^*(s)\,|$$

must be large in comparison with some angular velocity associated with the amplitude in question.

8·42. *Simplification of the conditions*

Although the three stability Conditions A, B and C obtained in § 8·41 are suitable for direct application to determine whether the motion of any given rocket is stable, they involve so many parameters of different kinds that it is difficult to ascertain what are the most important factors govening stability. In this subsection we introduce various simplifications in order to obtain a clearer picture of the meaning of the conditions. We shall not confine our remarks to the burning period of the rocket, but may occasionally consider any arc of the trajectory over which the angular deviation from the tangent at one end point remains small. It is easy to see that the equations continue to hold over any such arc with only minor modifications. After burnt the parameters κ and f $(=-f_R)$ must, of course, be omitted. (For R/m is small in comparison with $f_Q = QW/m$.)

The first simplification we can make is to omit from $G_1(s)$ and $G_2(s)$ those terms which are believed to be of smaller order than the others. Also we put $\epsilon = 0$ and omit the equality sign in Condition A in order to increase the stringency of the condition. The functions $G_1(s)$ and $G_2(s)$ are defined by equations (4·2·18, 19). We shall omit the terms κ^2, $d\kappa/dt$ and $(d\alpha_1/dt)\,V$ from $G_1(s)$ and $d\beta_1/dt$ from (4·2·19). Further we omit the two terms

$$\beta_1 \frac{f}{V} - \frac{\beta_1}{r}\frac{dr}{dt}$$

from $G_2(s)$. For they can be written in the form

$$-\beta_1 V r^{-1} \frac{d}{dt}\left(\frac{r}{V}\right)$$

during burning, and this is small when r/V is a slowly varying function of the time, as will usually be the case.† After burnt the two terms can be neglected since f can be omitted, and the rate of decrease of spin is entirely due to the aerodynamic couple and is small.

We then have

$$G_1(s) = n_1^2 V^2 + \beta_1^2 r^2 + 2\kappa\alpha_1 V + \kappa \frac{f}{V}, \tag{8·42·1}$$

and

$$G_2(s) = \varpi_1 V - 2\kappa\beta_1. \tag{8·42·2}$$

Condition A now becomes, by (8·41·8),

$$S = 4\left\{\alpha_2 V + \kappa + \frac{f}{V}\right\}^2 \left[n_1^2 V^2 + \beta_1^2 r^2 + 2\kappa\alpha_1 V + \kappa\frac{f}{V} + \left\{\alpha_2 V + \kappa + \frac{f}{V}\right\}^2\right] - (\varpi_1 V - 2\kappa\beta_1)^2 r^2 > 0. \tag{8·42·3}$$

Before reducing this further it is worth pointing out that S is an increasing function of f/V and, provided that ϖ_1 is not too great and negative, is also an increasing function of κ. This may also be deduced directly from (8·41·6). From it we conclude that *acceleration damping* (i.e. the effect of the terms in f/V) and *jet damping* (terms in κ) have a stabilizing influence on the motion of the rocket. This explains the well-known fact frequently observed in trials that many rockets which become unstable after burnt have a comparatively steady motion during burning.‡

Now write

$$S_1 = 4G_4^2\beta_1^2 - (\varpi_1 V - 2\kappa\beta_1)^2, \tag{8·42·4}$$

and

$$S_2 = -4G_4^2\left\{n_1^2 V^2 + 2\kappa\alpha_1 V + \kappa\frac{f}{V} + G_4^2\right\}. \tag{8·42·5}$$

† E.g. when the projectile is spun by inclined jets. The neglect of the two terms may not be justifiable immediately after launch in the case of a constant or a decreasing spin; in view of the marked effect of acceleration and jet damping in the early stages of flight, this however is not likely to be important.

‡ There are, of course, several other factors which contribute to the decrease in stability after burnt. Among them are the effects of supersonic velocity and of accelerated motion upon the position of the centre of pressure of the lift moment.

Then (8·42·3) may be written in the form

$$S_1 r^2 > S_2. \tag{8·42·6}$$

Accordingly, Condition A for stability will be satisfied in the following three cases:

Case A (i): $S_1 > 0, \quad S_2 \geqslant 0, \quad |r| > [S_2/S_1]^{\frac{1}{2}}.$

Case A (ii): $S_1 \geqslant 0, \quad S_2 < 0.$

Case A (iii): $S_1 < 0, \quad S_2 < 0, \quad |r| < [S_2/S_1]^{\frac{1}{2}}.$

After burnt these conditions may be simplified still further, since we can omit terms in f and κ and obtain

$$S_1 = V^2(4\alpha_2^2\beta_1^2 - \varpi_1^2) = V^2\left(\frac{2\beta_1 k}{m} + \varpi\right)\left(\frac{2\beta_1 k' d_2}{A} - \varpi\right),$$

and

$$S_2 = -4V^4\alpha_2^2(n^2 - \alpha_1^2 + \alpha_2^2) = -4V^4\alpha_2^2\left\{n^2 + \frac{kk' d_2}{mA}\right\},$$

so that, in cases A (i) and (iii)

$$\left(\frac{S_2}{S_1}\right)^{\frac{1}{2}} = S_3 = 2\,|\,\alpha_2\,|\,V\left[\frac{-\left(n^2 + \dfrac{kk' d_2}{mA}\right)}{\left(\dfrac{2\beta_1 k}{m} + \varpi\right)\left(\dfrac{2\beta_1 k' d_2}{A} - \varpi\right)}\right]^{\frac{1}{2}}. \tag{8·42·7}$$

Normally both k and $k'd_2$ will be positive, and when this is so, cases A (i), (ii) and (iii) become:

Case A (i): $n^2 \leqslant -\dfrac{kk' d_2}{mA}, \quad -\dfrac{2\beta_1 k}{m} < \varpi < \dfrac{2\beta_1 k' d_2}{A}, \quad |r| > S_3.$

The first two inequalities will be satisfied if the lift moment is destabilizing and if the Magnus moment is sufficiently small.

Case A (ii): $-\dfrac{2\beta_1 k}{m} \leqslant \varpi \leqslant \dfrac{2\beta_1 k' d_2}{A}, \quad n^2 > -\dfrac{kk' d_2}{mA}.$

These inequalities will hold when the Magnus moment is sufficiently small and the lift moment is stabilizing, or is sufficiently small and destabilizing. We have assumed here that $kk'd_2/mA$ is not negative as will usually be the case. The modifications necessary when this is not so are easily made.

Case A (iii): $\varpi < -\dfrac{2\beta_1 k}{m}$ or $\varpi > \dfrac{2\beta_1 k' d_2}{A}$, and $n^2 > -\dfrac{kk' d_2}{mA}, \quad |r| < S_3.$

This condition shows that when the Magnus moment is large stability can only be attained if the lift moment is stabilizing (or sufficiently small and destabilizing) and the spin is sufficiently small.

Conditions A (ii) and A (iii) combined show, in particular, that for an unrotated rocket to be stable it is necessary that

$$n^2 > -\frac{kk' d_2}{mA}.$$

The quantity $kk'd_2/(mA)$ is very small, so that for practical purposes it can be replaced by zero in the above three cases. Further, it is likely that β_1 and β_2 may each be replaced by β without appreciable error in most practical cases.

Before passing on to the consideration of Conditions B and C, it may be remarked that the stability condition usually employed in practice—and, in fact, the only one in regular use—has not appeared in the preceding analysis. This condition may be stated in the form

$$\beta^2 r^2 + n^2 V^2 = \beta^2 r^2 - \nu^2 V^2 > 0, \tag{8·42·8}$$

and is obtained when all damping and Magnus coefficients are neglected. For then $G(s)$ is real and equal to $n^2 + \beta^2 r^2 / V^2$, and this clearly must be positive if the yaw equation (4·3·8) is to have stable solutions. Alternatively we may deduce (8·42·8) from (8·41·6). For if we put $\epsilon = 0$ and neglect all damping and Magnus terms, we obtain $G_2 = 0$, $G_3 = G_1$, and hence, since $G(s) \neq 0$,

$$G_1(s) = n^2 V^2 + \beta^2 r^2 > 0.$$

We now consider Condition B of § 8·41. The expression (8·41·13) is of the order of magnitude of the angular deviation due to gravity,† and is therefore small during burning, by our assumptions. Accordingly Condition B does not apply to the motion of a rocket during burning, and, in fact, will only be of importance over arcs of the trajectory over which the gravity drop is appreciable. For such arcs the assumptions regarding the smallness of Z are, of course, not valid, and we are therefore not strictly justified in applying the formulae of § 8·2 in order to examine stability. The modifications which are necessary in order to make the theory applicable to the whole trajectory have, however, only a minor effect upon the stability conditions, since the yaw will still be expressible as a sum of parts $\Xi^{(1)}$, $\Xi_{\text{I}}^{(2)}$ and $\Xi_{\text{II}}^{(2)}$, and $\Xi_{\text{I}}^{(2)}$ will still be of the form (8·41·11). The functions $k_1(u, s)$ and $k_2(u, s)$, although still slowly varying or monotonic, will, of course, be no longer independent of Θ and Ψ (see, for example, Chapter 12 of Kelley, McShane & Reno (1949?)). It will therefore be possible to draw similar conclusions regarding the stability of the motion.

Accordingly, we proceed to examine Condition B in the case when the gravity drop is not small. We omit terms in κ and f since we need only consider the motion after burnt. Then Condition B states that

$$|\beta_2 r| - \{\tfrac{1}{2}(G_3 + G_1)\}^{\frac{1}{2}} = \frac{Kg\cos\alpha}{V} \tag{8·42·9}$$

where $|K|$ is large. It follows that

$$G_3 + G_1 = 2\left\{|\beta_2 r| - \frac{Kg\cos\alpha}{V}\right\}^2,$$

and hence
$$r^2 G_2^2 = 4\left\{|\beta_2 r| - \frac{Kg\cos\alpha}{V}\right\}^2 \left[\left\{|\beta_2 r| - \frac{Kg\cos\alpha}{V}\right\}^2 - G_1\right]. \tag{8·42·10}$$

If we can neglect Magnus and damping terms in comparison with terms in r, n^2 and K (8·42·10) simplifies considerably. For we then have

$$G_3 = |G_1| = G_1 = V^2 n^2 + \beta^2 r^2,$$

since, by (8·42·8), G_1 must be positive, and then

$$n^2 = -\frac{2|\beta r|\, Kg\cos\alpha}{V^3}\left(1 - \frac{Kg\cos\alpha}{2|\beta r|\, V}\right).$$

† This is obvious if 'infinite stability' ($n^2 = \infty$) is assumed, and is in fact true whenever Condition A is satisfied.

Also $Kg\cos\alpha < |\beta r|\, V$ by (8·42·9), and therefore

$$|n^2| > \frac{|K\beta r|\, g\cos\alpha}{V^3}. \qquad (8\cdot42\cdot11)$$

Accordingly, in this case, *Condition B for stability states that n^2 must be large in absolute magnitude in comparison with*

$$\frac{|\beta r|\, g\cos\alpha}{V^3}$$

if the gravity drop over the part of the trajectory under consideration is appreciable.

The type of instability which ensues when Condition B is not satisfied may be called *non-precessional instability*, since it occurs when one of the two modes of precession is of very low frequency. It is also known by the more special names of *vertex instability* or *overstability* since it is clear that $|\beta r|\, g\cos\alpha / V^3$ will be large (*a*) near the vertex when the velocity V is least and $\cos\alpha \doteqdot 1$, and (*b*) when the spin is very great. In fact

$$\frac{2\,|\beta r|\, g}{|n^2|\, V^3}$$

is the so-called *vertex yaw*.

We now examine the Condition C for stability. Just as in the case of Condition B, this condition will not be important except over long distances, since otherwise

$$\int_{s_0}^{s} \omega_T\, dt_u$$

will remain small. We shall therefore suppose that this quantity is not small. This means that α_C is the only tolerance which we need consider, since the other five cease at burnt. As mentioned before, the solution obtained in § 8·2 does not apply unless the angular deviation of the trajectory remains small. The part $\Xi^{(2)}_{\mathrm{II}}$ of the yaw will, however, still be given by an expression of the form (8·41·14), and Condition C for stability may accordingly be put in the form

$$r^2 G_2^2 = 4\{|(1-\beta_2)\, r| - K'\omega_T\}^2\,[\{|(1-\beta_2)\, r| - K'\omega_T\}^2 - G_1], \qquad (8\cdot42\cdot12)$$

where $|K'|$ is large in comparison with unity.

When Magnus and damping terms are omitted, this reduces to

$$n^2V^2 = r^2(1-2\beta) - 2\,|r|\,(1-\beta)\, K'\omega_T + K'^2\omega_T.$$

Now, by (4·2·22), (5·2·4, 5) and (8·2·13),

$$\omega_T \doteqdot |\mu_3| = (1-2\beta)\,|r|\,\alpha_C.$$

Hence we have

$$\begin{aligned} n^2V^2 &\doteqdot r^2(1-2\beta)\,[1 - 2(1-\beta)\, K'\alpha_C + K'\alpha_C^2(1-2\beta)] \\ &= r^2(1-2\beta)\,(1-K'\alpha_C)\,\{1-(1-2\beta)\, K'\alpha_C\} \\ &\doteqdot r^2(1-K'\alpha_C)^2. \end{aligned}$$

We therefore conclude that *Condition C for stability will be satisfied except when*

$$\int_{t_0}^{t} r\alpha_C\, dt$$

is not small, and $n^2V^2 = r^2$ approximately.

The type of instability which ensues when Condition C is not satisfied may be called *resonance instability*. It is clear that such instability can only occur when the lift moment is stabilizing and the spin-velocity ratio is extremely low.

8·43. *Summary*

We summarize here, in more general terms, the conclusions which can be drawn from the preceding mathematical analysis.

We have found that if the motion of a rocket along its trajectory is to be stable three important conditions, which we have called A, B and C, must be satisfied. Corresponding to these three conditions there are three main types of unstable motion which can arise when the appropriate condition is violated. It is convenient to call them instability of type A, B and C respectively, and we consider them individually below.

Instability of type A. This type of instability is easily detectable in flight since the yaw builds up rapidly when Condition A is not satisfied. In the case of rockets the instability may not appear till after burning has ceased, since the acceleration of the projectile, and the gases ejected at the rear end, exert a damping effect upon the yaw. For this reason the conditions A (i), (ii) and (iii) of § 8·42 are likely to be more stringent after the end of burning. Accordingly, when a rocket is designed which is to have a useful trajectory beyond the end of burning, the stability criteria should first be applied with the values of the parameters appropriate to the end of burning.† With shell, on the other hand, instability of type A is most likely to appear immediately after the shell has left the gun barrel, and for this reason is often called *muzzle instability*.

Condition A depends in a complicated way upon the aerodynamic force and couple coefficients, and for this reason instability can manifest itself in a number of different circumstances which, to the layman, might appear to have very little in common. The two most important factors are the lift moment $\mathbf{M}_1$ and the Magnus moment $\mathbf{M}_3$. Their effects may be described broadly as follows.

(*a*) If the lift moment is destabilizing (i.e. if the associated centre of pressure lies ahead of the centre of gravity) the motion will be unstable unless the Magnus moment is small and, at the same time, the spin is sufficiently large. If the Magnus moment is large, instability is unavoidable (Case A (i)).

(*b*) If the lift moment is stabilizing and the Magnus moment is sufficiently small, the motion will be stable, whatever the magnitude of the spin may be (Case A (ii)).

(*c*) If the lift moment is stabilizing and the Magnus moment is not small, the motion will be unstable unless the spin is sufficiently low (Case A (iii)).

These statements are based upon the form of the stability conditions after burnt. For a more detailed and qualitative account the reader is referred to subsections §§ 8·41, 8·42.

The type of instability which occurs when the lift moment is destabilizing is the most common in practice, and is the only type referred to in most accounts. Unfortunately, owing to lack of information on the magnitudes of the coefficients of the damping and Magnus forces, it is not possible in practice to apply the inequalities occurring in the Cases A (i), (ii) and (iii) of § 8·42, and for this reason it is usual to employ the less accurate condition (8·42·8) alone, and to ensure that it is satisfied by a comfortable margin.

† These remarks do not, of course, apply to fire from aircraft.

The type of instability mentioned in (*c*) above has been observed with certain experimental rockets (fast spinning projectiles with straight fins and large length-diameter ratio), and, in fact, the inequalities of Case A (iii) have been used in order to obtain rough information on the magnitude of the Magnus moment.

Instability of type B. This type of instability, which may be called *non-precessional instability*, is unlikely to appear during the burning period of a rocket since it builds up very slowly over long distances. It occurs when one of the modes of precession is of very low frequency so that the axis of the projectile, instead of precessing round the tangent to the trajectory, tends to stay pointed in the same direction or else precesses only very slowly. As would be expected, it is near the vertex of the trajectory, where the curvature is greatest, that this type of instability generally appears, especially if the Q.E. is high. Non-precessional instability will occur ultimately with every projectile if the axial spin is increased indefinitely. This implies that there is an upper limit to the amount of spin which it is desirable to give a projectile. Since, in case (*a*) above, there is also a lower limit, it follows that in many cases it may require considerable care to decide the optimum amount of spin to impart.

Instability of type C. This type of instability can only occur when the projectile has a considerable degree of axial asymmetry, for example, if the displacement of the longitudinal principal axis of inertia is appreciable.† Instability will then occur if the lift moment is stabilizing and if the lateral yawing motion resonates with the axial rotation, i.e. if the spin is low and $n^2V^2 = r^2$ approximately. Accordingly, we may call it *resonance instability*. It builds up slowly over long distances in the same manner as non-precessional instability, and is therefore unlikely to occur during burning.

8·5. Further remarks

The investigation of stability carried out in § 8·4 has been based upon the general solution of the equations of motion which was derived in § 8·2. This solution is an approximate one which holds when the function $G(s)$ possesses the properties (*a*) to (*d*) of that section. It may therefore be of value to indicate how the procedure may be modified when these properties, and in particular (*b*), (*c*) and (*d*), are not possessed by the function $G(s)$.

In the first place, suppose that $G(s)$ vanishes at some point of the interval. In practice $G(s)$ is, of course, unlikely to vanish identically since it is a complex function of s. It is of value, however, to consider this case since it is of importance when the approximation $G(s) = n^2+\beta^2r^2/V^2$ is made. If n^2 is negative, $G(s)$ may increase through zero from negative values to positive values as the velocity increases. The solutions obtained will clearly not hold near a zero of $G(s)$ since $G(s)$ occurs in the denominator in (8·2·8). If, however, $G(s)$ has a zero of order m at $s = u_0$ say, it is possible to approximate to the solution of the differential equation by means of Bessel functions of order $1/(m+2)$ near u_0 and by this means bridge the gap between the solutions valid for values of s on either side of the zero (Goldstein 1928).

† This is the only tolerance which has been considered in the mathematical analysis. The same phenomenon, however, may occur if the rocket has fins which are bent or damaged, as this may cause a malalined torque and set up a slow rotation which may ultimately resonate with the lateral yawing motion.

This case is of particular importance in connexion with the projection of jet-rotated finless rockets from aircraft, since then $n^2+\beta^2r^2/V^2$ will be of the form

$$G(s) = -\nu^2+\beta^2\gamma^2\left(1-\frac{U}{V}\right)^2,$$

where U is the aircraft speed. In practice it is usually not possible to make γ sufficiently large so that this expression is always positive, and therefore the condition (8·42·8) for stability is not satisfied in the early stages of flight. It should perhaps be mentioned that it is doubtful whether the above approximation to $G(s)$ is valid in this case, since, even if Magnus forces and jet and aerodynamic damping forces are negligible, the terms $\beta f/V$ and $-\beta(dr/dt)/r$ appearing in (4·2·19) no longer cancel each other out, since the spin is not proportional to the total velocity. A better approximation is

$$G(s) = -\nu^2+\beta^2\gamma^2\left(1-\frac{U}{V}\right)^2+i\beta\gamma\frac{fU}{V^3}, \tag{8·5·1}$$

and then conditions (b) and (c) are both satisfied. The solution obtained in § 8·2 can therefore be used provided that the condition (d) is satisfied. In this particular case it is, however, to be expected that the function defined in (8·5·1) will not possess the property (d) since its imaginary part is a rapidly diminishing function of the velocity.

Accordingly, we now consider the general case when $G(s)$ possesses the properties (a), (b) and (c) but not (d). We then have to carry the approximation of § 8·2 further by putting

$$\eta = G^{\frac{1}{2}}+\tfrac{1}{4}iG'G^{-1}+\epsilon_2,$$

and to determine ϵ_2 from the equation

$$i\eta'-\eta^2+G = 0.$$

This will, of course, make the solution more complicated, particularly if further terms ϵ_3, ϵ_4, ... are necessary (see Wentzel 1926). In general it may be concluded that if terms ϵ_2, ϵ_3, ... require to be taken into account, this method is not likely to be of much use, and, if the equation cannot be transformed into a form more amenable to approximate treatment, numerical methods or else a differential analyzer should be used. In the case when $G(s)$ is given by (8·5·1) this last course is to be preferred, since the form of the motion can be extremely sensitive to small variations in ν^2, γ and U and approximate treatments of the general case are therefore likely to be considerably in error.†

In conclusion it may be remarked that the condition (c) on $G(s)$ is not a vital one. The analysis may be carried out as before when it is not satisfied. It will, of course, be necessary to keep track of the variation of the argument of $G(s)$ in order that the functions $[G(s)]^{\frac{1}{2}}$ and $[G(s)]^{\frac{1}{4}}$ may be defined properly. For example, the function $g_1^*(s)$ may take negative values along some parts of the trajectory.

8·6. Historical note on the stability conditions

The definition of stability which we have used in § 8·4 and the procedure which we have adopted to find the three stability conditions A, B and C differ somewhat from those employed by other writers. Thus Condition B is generally not mentioned in chapters dealing with

† With the help of the Manchester differential analyzer about 100 solutions of the yaw and deviation equations, corresponding to different values of the parameters ν^2, γ, U and f were run off in about a fortnight, and form an extremely valuable set of data.

stability, but appears in those chapters which are concerned with drift and the 'yaw of repose', while Condition A, or some form of it, is usually derived by substituting $\Xi = A\,e^{iqt}$ in the equation for the yaw (whose coefficients are assumed to be slowly varying or constant functions) and determining under what conditions q is real. The method of § 8·4 has been preferred, particularly in view of the highly oscillatory character of the function $R(s)$, and because of the large variation in velocity during burning.

The Conditions A, B and C have been based on the stability conditions obtained earlier (Rankin (1943 *a*, § 3·4)) for the particular case of spin proportional to velocity. Condition A of § 8·41 is identical with that obtained in Chapter 12 of Kelley, McShane & Reno (1949?, first issued as a report by Kelley & McShane 1944), except that they include terms involving $\sin\Theta$ which are probably of smaller order than the remaining terms, and are negligible during the burning period of the rocket because of our assumptions. Their Cases I and II correspond exactly to our cases A (i) and A (ii), (iii) respectively, our parameters α_2 and S_1 being essentially equivalent to their s_1 and $s_2 s_3$. Similar relations were also given earlier by Nielsen & Synge (1946, first issued 1943).

The earliest occurrence of a stability condition of a type more stringent than the classic 'spinning-top condition' (8·42·8) that we can trace is that given by Fowler *et al.* in the *Text-book of anti-aircraft gunnery* (1925). The condition given there is implicit in their earlier paper (1920), and is a slightly weaker form of our Condition A (i) in which the lift, lift moment, Magnus moment M_3 and damping moment M_2 are the only forces and couples included.

The Condition C does not appear to have been stated before in mathematical terms, probably because it applies only to asymmetric projectiles with a stabilizing lift moment.

9. THE EVALUATION OF THE PARAMETERS

9·1. General

In order to apply the mathematical solutions of the equations of motion to any particular rocket, it is necessary to know the numerical values of the parameters which occur. These parameters fall roughly into six groups:

(i) The velocity V and the distance travelled s at each instant.

(ii) The axial spin.

(iii) The initial conditions at launch from the projector (see § 3·7).

(iv) Parameters directly connected with the consumption of the charge during burning, such as the mass m, the moments of inertia A, C, the distance l, the damping parameter κ and the ratios β, β_1 and β_2.

(v) Parameters depending upon the aerodynamic coefficients such as n^2, α_1 and α_2.

(vi) Parameters depending upon asymmetries of design and functioning and on the manufacturing 'tolerances', e.g. α_R, ϕ_R, etc.

These are not rigid divisions, as there is a certain amount of overlapping between the groups. Thus the aerodynamic parameter n^2 depends also upon the velocity and the transverse moment of inertia A.

In this section it is not proposed to give a complete account of the methods which can be used to determine all the parameters involved. Such an account, if comprehensive, would be of a highly technical nature and would take up a considerable amount of space. Thus, for example, the estimation of the parameters included in groups (i) and (v) depends upon an accumulation of experimental data (static firing tests, photographic observation, wind tunnel tests, etc.), so that the methods used, which are often of an empirical nature, are not fixed but are constantly being modified and improved as new data become available. We shall confine our attention to those parameters whose form and variation during burning can be deduced from the preceding mathematical analysis, i.e. to the parameters occurring in groups (ii) and (iv). For certain designs of projector it is possible to extend the mathematical theory to cover motion on the projector and so obtain estimates of the parameters in group (iii). It has, however, been found that the values of the initial yaw, initial rate of turn, etc., obtained in this way are in many cases appreciably less than those deduced from the analysis of photographic records of actual projectiles in flight;† further, it is necessary to make assumptions regarding the character of the motion on the projector which may not be warranted. For these reasons we do not consider the third group. The sixth group of parameters is one which, although not absent in other missiles, is of especial importance for the rocket, since it forms the major cause underlying the large dispersion which is so characteristic of the weapon. For this reason, we discuss briefly in § 9·8 the problem of determining the parameters in group (vi), but do not consider in detail any of the practical methods which can be used for this purpose.‡

In §§ 9·2 to 9·6 we shall assume that the velocity, or else the total mass, is known at each instant.

9·2. Parameters connected with the burning of the charge

The decrease in mass during burning affects the position of the centre of gravity and the moments of inertia. We give here formulae from which the values of l, A, C and other parameters can be calculated at any instant when the total mass at any instant is known. These formulae apply to certain simple forms of charge which are commonly used, and are valid when the rate of burning is uniform at all points of the charge surface. Similar formulae may be derived for other types of charge.

The quantities l, m, A, C have been defined with respect to all the matter included within the outer surface of the rocket and within the exit plane, i.e. the burning gases in the interior of the projectile are included as well as the solid components. Since the mass of these gases is very small, it is legitimate to neglect them when evaluating the parameters.

We consider two types of charge. Each type consists of a number N_c of similar tubular sticks of length l_c, internal radius r_c and external radius R_c.

Type 1. The sticks are all symmetrically disposed with their axes lying on a cylinder of radius ρ_c whose axis passes through the charge centre G_1 and is approximately parallel to

† This discrepancy is really not surprising when it is considered that the effects of blast from the burning gases ejected from the rocket, and of friction, during the motion on the projector, are largely unknown, and cannot therefore be adequately taken into account in the mathematical theory.

‡ The Ministry of Supply 'monograph', upon the mathematical part of which this paper is based, contains chapters devoted to the estimation of all the parameters in groups (i) to (vi).

the rocket's axis OC_0. When $\rho_c > 0$ we assume that $N_c \geqslant 3$. When $\rho_c = 0$, $N_c = 1$, i.e. the charge consists of a single stick.

Type 2. This is similar to type 1 ($\rho_c > 0$), but there is an additional central stick of the same dimensions. To avoid overlapping we must have

$$\rho_c \geqslant 2R_c.$$

We assume that $N_c \geqslant 4$.

In the most usual arrangement of type 2, $N_c = 7$ and $\rho_c = 2R_c$, i.e. six sticks of charge are closely grouped about one central stick.

The restrictions on N_c are necessary in order that the symmetry may be of order greater than 2.

Write

$$\vartheta = \begin{cases} 1 & \text{charge of type 1,} \\ 1 - \dfrac{1}{N_c} & \text{charge of type 2,} \end{cases} \tag{9·2·1}$$

and put

$$\kappa_A^2 = \tfrac{1}{12} l_c^2 + \tfrac{1}{4}(R_c^2 + r_c^2) + \tfrac{1}{2}\vartheta \rho_c^2, \tag{9·2·2}$$

$$\kappa_C^2 = \tfrac{1}{2}(R_c^2 + r_c^2) + \vartheta \rho_c^2. \tag{9·2·3}$$

When $\rho_c = 0$, κ_A and κ_C are respectively the transverse and axial radii of gyration of a single unburnt stick of charge about its centre of gravity. Write c for the mass of charge unburnt at any instant. Then

$$c = m - m_1. \tag{9·2·4}$$

It is easy to show that, at any instant during burning,

$$l = l_{00} + \frac{q}{m}(m_{00} - m) = q_1 + q\frac{m_{00}}{m}, \tag{9·2·5}$$

$$A = A_{00} - (m_{00} - m)\left\{\kappa_A^2 + q^2\frac{m_{00}}{m} + \frac{c(c_{00}+c)}{8c_{00}^2}(R_c - r_c)^2\right\}, \tag{9·2·6}$$

and

$$C = C_{00} - (m_{00} - m)\left\{\kappa_C^2 + \frac{c(c_{00}+c)}{4c_{00}^2}(R_c - r_c)^2\right\}. \tag{9·2·7}$$

These formulae are valid for both types of charge provided that the solid charge does not move about in the body during burning, and if no burning takes place on the flat ends of the charge sticks. They may be used even when the flat ends form part of the burning surface provided that $R_c - r_c$ is small in comparison with l_c. This condition will usually be satisfied in practical cases. Since

$$\frac{c(c_{00}+c)}{8c_{00}^2}(R_c - r_c)^2$$

will usually be quite small in comparison with $\frac{1}{12}l_c^2$ and therefore with κ_A^2, we may write

$$A \doteqdot A_{00} - (m_{00} - m)\left\{\kappa_A^2 + q^2\frac{m_{00}}{m}\right\}. \tag{9·2·8}$$

We shall also write

$$C \doteqdot C_{00} - (m_{00} - m)\,\kappa_C^2. \tag{9·2·9}$$

Both (9·2·8) and (9·2·9) are exactly correct when $t = 0$ or $t = t_1$. The maximum errors occur, in either case, when $c^2 = \frac{1}{3}c_{00}^2$, and are

$$\frac{c_{00}(R_c - r_c)^2}{12\sqrt{3}} \quad \text{and} \quad \frac{c_{00}(R_c - r_c)^2}{6\sqrt{3}}$$

in the two cases. Both these amounts are usually quite insignificant.

Yaw damping. In order to evaluate the damping parameter κ of (4·2·7) it is necessary to know dA/dt. We have, from (9·2·6),

$$\frac{dA}{dt} = -Q\left\{\kappa_A^2 + q^2\frac{m_{00}^2}{m^2} - \frac{(c_{00}-c)(c_{00}+2c)}{2c_{00}^2}(R_c - r_c)^2\right\} \tag{9·2·10}$$

$$\doteqdot -Q\left\{\kappa_A^2 + q^2\frac{m_{00}^2}{m^2}\right\}. \tag{9·2·11}$$

It follows that

$$\kappa = \frac{Q}{2A}\left\{l^2 - \kappa_A^2 - q^2\frac{m_{00}^2}{m^2}\right\}. \tag{9·2·12}$$

The integral of κ is required in order to evaluate the damping function $P(s)$. We have

$$\int_{t_0}^{t}\kappa\,dt = \frac{q_1^2-\kappa_A^2}{4\kappa_A^2}\log\frac{m_0A_0}{mA} - \frac{b(q_1^2-\kappa_A^2)-4qq_1\kappa_A^2}{2\kappa_A^2(b^2+4\kappa_A^2q)^{\frac{1}{2}}} \times \coth^{-1}\left\{\frac{2\kappa_A^2mm_{00}+bm_0(m+m_0)-2q^2m_0m_{00}}{m_0(m_0-m)(b^2+4\kappa_A^2q)^{\frac{1}{2}}}\right\}, \tag{9·2·13}$$

where

$$b = \frac{A_{00}}{m_{00}} - \kappa_A^2 + q^2 \tag{9·2·14}$$

Since this formula is not a simple one, it may be preferable to obtain this integral by numerical integration from (9·2·12).

Spin damping. The damping effect of the jet upon the axial spin can be determined when the integral

$$\int_0^t \frac{Q\,dt'}{C'}$$

is known. We have, by (9·2·9),

$$\int_0^t \frac{Q\,dt'}{C'} = \frac{1}{\kappa_C^2}\log\frac{C_{00}}{C} = \frac{m_{00}-m_1}{C_{00}-C_1}\log\frac{C_{00}}{C}. \tag{9·2·15}$$

Displacement of the longitudinal principal axis of inertia. In § 6·102 the quantity $m(dA/dt)/QWA$ occurs. By (9·2·11), we have

$$-\frac{m(dA/dt)}{QWA} = \frac{m\kappa_A^2 + q^2m_{00}^2/m}{WA}, \tag{9·2·16}$$

where A is given by (9·2·8).

In all the formulae given above the mass m has been the independent variable. If desired, the various quantities can be expressed in terms of the velocity V by means of (4·4·3).

9·3. The form of the axial spin

Spin can be imparted to a rocket in a number of different ways. We consider here two methods of spinning a rocket which we shall call jet rotation and fin rotation. In jet-rotated projectiles an axial torque is obtained by causing the burning gases (or a proportion of them) to escape through a system of inclined nozzles. A variety of different designs of multiple nozzle systems can be used (see § 9·4). By fin rotation we mean any method in which the fins are used directly in order to provide a torque. For example, if the blades are offset, such a couple will be produced by the action of the aerodynamic forces in flight (cf. § 3·53). Alternatively, the spin may be imparted by causing the fins (or other bearing surfaces) to slide against spiral rails while on the projector. Such methods are considered in §§ 9·5 and 9·6.

In practice it may be found convenient to employ several methods of rotation at the same time. In such cases the best results are usually obtained by distributing the amount of rotation required among the different methods according to definite ratios (see § 9·5).

As would be expected, the spin during flight can be reduced considerably if the projectile has large fins, unless they are offset by a suitable amount.

We start from the general expressions for the spin given in formulae (4·4·4, 5). We shall suppose throughout that the spin r_{00} at the instant of ignition is zero. It follows that

$$r = \frac{1}{C}\int_0^t (G_R + \Gamma_F V^2 \Delta_F) \exp\left\{-\int_{t'}^t (\Gamma Va + Qk_e^2)\frac{dt''}{C}\right\} dt', \tag{9·3·1}$$

in the particular case when no additional couple acts before launch. When such a couple does act (for example, in the case of launch from a spiral projector) formula (4·4·4) must be used. In the integrals occurring in these formulae $G_R + \Gamma_F V^2 \Delta_F$ is a function of the time t' and $(\Gamma Va + Qk_e^2)/C$ is a function of the time t''.

In order to find r it is therefore necessary to know the values and variation of a number of quantities such as C, G_R, Γ_F, Γ_A and Q during burning. We assume that the rate of burning Q is known at each instant; the axial moment of inertia C can then be found when the disposition of the charge is known (see § 9·2). Also the aerodynamic parameters Γ_F and Γ_A are assumed to be known. The jet torque $\mathbf{G}_R$ can be found, when the form of the nozzle system and the charge characteristics are known, by the methods given in the following subsection.

The term
$$\exp\left\{-\int_{t'}^t Qk_e^2 \frac{dt''}{C}\right\}$$
is called the *jet damping factor*. It can be evaluated when the form of the charge is known. We have, in fact, from (9·2·15)

$$\int_0^t \frac{Qk_e^2}{C}\, dt'' = \frac{k_e^2}{\kappa_C^2} \log \frac{C_{00}}{C}, \tag{9·3·2}$$

where κ_C, the axial radius of gyration of the unburnt charge, is given by (9·2·3).

$$\exp\left\{-\int_{t'}^t \Gamma_A Va \frac{dt''}{C}\right\} = \exp\left\{-\int_{s'}^s \frac{\Gamma_A a}{C}\, ds''\right\} = e^{-\tau(s',\, s)} \tag{9·3·3}$$

is called the *aerodynamic damping factor*.

It follows from (9·3·2, 3) that the total damping factor is

$$\exp\left\{-\int_{t'}^t (\Gamma_A Va + Qk_e^2)\frac{dt''}{C}\right\} = \left(\frac{C'}{C}\right)^{-\nu} e^{-\tau(s',\, s)}, \tag{9·3·4}$$

where
$$\nu = k_e^2/\kappa_C^2. \tag{9·3·5}$$

9·4. Spin imparted by inclined nozzles

We consider the following general design of multiple nozzle. It is assumed that the same nozzle system is used for propelling the projectile forward as for imparting the spin.

9·41. *The nozzle system*

Suppose that the nozzle system consists of N concentric rings of similar nozzles with say n_p equally spaced nozzles in the pth ring. There may also be a central nozzle which is not inclined; put $n_0 = 0$ or 1 according as there is not or is such a central nozzle. Consider any nozzle of the pth ring ($p \geqslant 0$). Let its central axis be a distance a_p from the axis of the projectile, and let Δ_p be the angle between them. Let $2b_p$ be the diameter of the nozzle (perpendicular to the nozzle's central axis) at the exit plane. Then, if the amount of gas emitted from each nozzle is proportional to its cross-section† (i.e. proportional to b_p^2), and if, for each nozzle, the resultant gas velocity W' is along the central axis of the nozzle ('aberration' due to spin being ignored), the total 'thrust' is approximately

$$QW = QW' \sum_{p=0}^{N} n_p b_p^2 \cos\Delta_p \Big/ \sum_{p=0}^{N} n_p b_p^2. \tag{9·41·1}$$

and the axial torque is

$$G_R = QW' \sum_{p=0}^{N} n_p a_p b_p^2 \sin\Delta_p \Big/ \sum_{p=0}^{N} n_p b_p^2. \tag{9·41·2}$$

We define Δ_e, the *effective nozzle inclination* and a_e the *effective pitch-circle radius* by

$$\cos\Delta_e = \sum_{p=0}^{N} n_p b_p^2 \cos\Delta_p \Big/ \sum_{p=0}^{N} n_p b_p^2, \tag{9·41·3}$$

and

$$a_e \sin\Delta_e = \sum_{p=0}^{N} n_p b_p^2 a_p \sin\Delta_p \Big/ \sum_{p=0}^{N} n_p b_p^2, \tag{9·41·4}$$

so that the thrust and torque are given by

$$QW = QW' \cos\Delta_e, \quad G_R = QW' a_e \sin\Delta_e. \tag{9·41·5}$$

It will be noted that the expressions for the thrust and torque do not take into account the velocity of the gases relative to the nozzles due to the spin. The effect of the spin is to diminish the angle of inclination Δ_p by approximately $a_p r/V$. In the case of the thrust this effect is negligible, since $a_p r/V$ is small by our Assumption A 1. In the case of the torque it has already been taken into account by the jet damping factor.

The effective radius of gyration k_e of the exit plane is given by‡

$$Qk_e^2 = \mathbf{OC}_0 . \int_{S_0} \rho v_N \mathbf{u} \times (\mathbf{\Omega} \times \mathbf{u})\, dS,$$

where S_0 is the total set of exit planes of the individual nozzles, and $\mathbf{u}$ is a vector from the centre of S_0. This formula holds even when the various exit planes are not coplanar, and gives

$$k_e^2 = \sum_{p=0}^{N} n_p b_p^2 \{a_p'^2 + \tfrac{1}{4} b_p^2 (1 + \cos^2\Delta_p)\} \Big/ \sum_{p=0}^{N} n_p b_p^2 \tag{9·41·6}$$

approximately, where a_p' is the distance of the centre of the exit plane of a nozzle in the pth ring from the axis. Clearly $a_p' \geqslant a_p$.

So far the relation between the radii a_p and the angles of offsetting Δ_p has been quite arbitrary. In order to obtain the maximum efficiency, however, it is desirable that there should be some relation between them when there is no more than one nozzle ring. Thus, although it may be possible to produce the desired torque in a variety of different ways it will generally be advisable to design the nozzle system so that the thrust QW is as great as possible. On comparing (9·41·1, 2) we see that this implies that

$$\frac{1}{a_1}\tan\Delta_1 = \frac{1}{a_2}\tan\Delta_2 = \ldots = \frac{1}{a_N}\tan\Delta_N, \tag{9·41·7}$$

when there is more than one ring of nozzles.

† It is assumed that the expansion ratio of exit area to throat area is the same for each nozzle.

‡ See § 2·5. For a symmetric projectile $\mathbf{k} = \mathbf{OK} = \mathbf{OC}_0$.

A particularly common case is that of a single ring of n_1 nozzles with or without a straight central nozzle of the same size. Then we have

$$\cos\Delta_e = \frac{n_0+n_1\cos\Delta_1}{n_0+n_1}, \tag{9·41·8}$$

$$a_e\sin\Delta_e = \frac{a_1 n_1}{n_0+n_1}\sin\Delta_1, \tag{9·41·9}$$

and

$$k_e^2 = \frac{n_1 a_1'^2}{n_0+n_1} + \frac{2n_0+n_1(1+\cos^2\Delta_1)}{4(n_0+n_1)}\,b_1^2. \tag{9·41·10}$$

In all cases the velocity is given by

$$V = W\log\frac{m_{00}}{m} = W'\cos\Delta_e\log\frac{m_{00}}{m}, \tag{9·41·11}$$

approximately, by (4·4·3), since $V_{00} = 0$.

9·42. *Rockets with small fins or no fins*

We assume that there are no fins or that the fins are not offset and are sufficiently small for the aerodynamic damping to be negligible. We then have, from (4·4·4), (9·3·4) and (9·41·5),

$$\begin{aligned} r &= \frac{Wa_e\tan\Delta_e}{C}\int_{t_0}^{t} Q\left(\frac{C'}{C}\right)^{-\nu} dt' + r_0\left(\frac{C_0}{C}\right)^{1-\nu} \\ &= \frac{Wa_e\tan\Delta_e}{(\kappa_C^2-k_e^2)}\left\{\left(\frac{C_0}{C}\right)^{1-\nu}-1\right\} + r_0\left(\frac{C_0}{C}\right)^{1-\nu}. \end{aligned} \tag{9·42·1}$$

When no additional torque is imposed before launch we can use (4·4·5) in place of (4·4·4) and then obtain

$$r = \frac{Va_e\tan\Delta_e}{(\kappa_C^2-k_e^2)\log(m_{00}/m)}\left\{\left(\frac{C_{00}}{C}\right)^{1-\nu}-1\right\}. \tag{9·42·2}$$

Equations (9·42·1, 2) hold provided that $\kappa_C \neq k_e$. When $\kappa_C = k_e$ we have

$$r = \frac{Wa_e\tan\Delta_e}{\kappa_C^2}\log\frac{C_0}{C} + r_0 \tag{9·42·3}$$

in place of (9·42·1), and

$$r = \frac{Va_e\tan\Delta_e}{\kappa_C^2}\,\frac{\log(C_{00}/C)}{\log(m_{00}/m)} \tag{9·42·4}$$

in place of (9·42·2).

Both equations (9·42·2, 4) may be written in the form

$$r = \gamma V, \tag{9·42·5}$$

where

$$\gamma = \frac{a_e\tan\Delta_e}{\kappa_C^2}\,\frac{\log(C_{00}/C)}{\log(m_{00}/m)}\left\{1+\frac{(\kappa_C^2-k_e^2)}{2\kappa_C^2}\log\frac{C_{00}}{C}+\dots\right\} \tag{9·42·6}$$

for $t>0$. At the instant $t = 0$

$$\gamma = \gamma_{00} = \frac{m_{00}a_e\tan\Delta_e}{C_{00}}. \tag{9·42·7}$$

For many designs of rocket the axial radius of gyration of the complete projectile does not alter appreciably during burning, and κ_C^2 and k_e^2 are usually approximately equal. It follows that, in these cases, γ remains approximately constant during burning. For some types of projectile, however (particularly those with light alloy motor tubes), the increase of the axial radius of gyration during burning may not be negligible, and then γ will usually decrease during burning. It is, of course, important that the correct value of γ should be used (i) when the stability is critical, and (ii) when the metal is highly stressed owing to a high rotation. The slight change in γ during burning is, however, not important (except in case (i)) when estimating dispersion, wind deviation, gravity drop, etc., since, as remarked elsewhere, it is in the early stages of flight that the greater part of the deviation of a rocket is built up. Thus, when using the formulae of §§ 5 and 6, the value γ_0 of γ at launch, or the value γ_{00} at ignition, may be taken.

9·43. *Rockets with large fins*

If the fin area is appreciable the aerodynamic damping may be considerable. The spin is given by

$$r=\frac{Wa_e\tan\Delta_e}{C}\int_{t_0}^{t}Q\,\mathrm{e}^{-\tau(s',\,s)}\left(\frac{C'}{C}\right)^{-\nu}dt'+r_0\,\mathrm{e}^{-\tau(s_0,\,s)}\left(\frac{C_0}{C}\right)^{1-\nu} \qquad (9\cdot43\cdot1)$$

when the fins are not offset. It is necessary to know the acceleration at each instant during burning in order to evaluate this integral. Even when constant acceleration is assumed (9·43·1) cannot be very much simplified unless the jet damping is negligible. If no additional couple acts while the projectile is on the projector (9·43·1) can be written as

$$r=\frac{Wa_e\tan\Delta_e}{C}\int_{0}^{t}Q\,\mathrm{e}^{-\tau(s',\,s)}\left(\frac{C'}{C}\right)^{-\nu}dt'. \qquad (9\cdot43\cdot2)$$

For a short distance after launch while $\tau(s_0, s)$ is small the analysis of § 9·42 can be applied, and (9·43·2) then gives

$$r=\gamma V=\frac{m_{00}a_eV\tan\Delta_e}{C_{00}}, \qquad (9\cdot43\cdot3)$$

approximately. This approximation cannot be used unless $\tau(s_0, s)$ is small.

9·5. Spin imparted by offset fins

We assume that each fin is offset by an amount Δ_F at its centre of pressure, as described in § 3·53, and consider the spin produced by this means alone. The coefficient Γ_F is assumed to be known and to be approximately constant (see § 3·53). Then, by (9·3·1),

$$r=\frac{\Gamma_F\Delta_F}{C}\int_{0}^{t}V^2\,\mathrm{e}^{-\tau(s',\,s)}\left(\frac{C'}{C}\right)^{-\nu}dt'. \qquad (9\cdot5\cdot1)$$

The angle Δ_F is assumed to be small. If this is not the case the assumptions made in § 3 do not hold, but a rough estimate may be obtained by replacing Δ_F by $\sin\Delta_F$ in (9·5·1).

The integral in (9·5·1) will, in general, have to be evaluated by step-by-step methods. If C does not vary appreciably during burning (9·5·1) becomes

$$r=\frac{\Gamma_F\Delta_F}{C_{00}}\int_{0}^{s}V\,\mathrm{e}^{-\tau(s',\,s)}\,ds'. \qquad (9\cdot5\cdot2)$$

During the early part of burning while $\tau(s_0, s)$ is small this gives

$$r=\frac{2\Gamma_F\Delta_F}{3C_{00}}V_s, \qquad (9\cdot5\cdot3)$$

approximately, if the acceleration is constant. Formula (9·5·3) may be a valid approximation throughout burning if the fins are small and if the time of burning is short. For rockets with longer times of burning it cannot be expected to hold except in the immediate vicinity of launch. For such rockets, especially when fitted with large fins, the aerodynamic damping becomes appreciable as the velocity increases, and the spin approaches the limiting case in which the relative wind at the centre of pressure of each fin is parallel to the fin surface. Thus during the later stages of burning

$$r = \frac{\Delta_F V}{a_F}, \tag{9·5·4}$$

approximately.

9·6. Spin imparted by a spiral projector

Suppose that the projector rails are set spirally so as to form a helix of constant pitch P. As the rocket moves forward bearing surfaces (usually the fins) slide against the rails and impart the spin. Let a_p be the radius of the helix, i.e. the radius to the point of each rail which is in contact with the bearing surface. The projectile will advance a distance $2\pi P$ with every revolution,† and its spin while on the rails is

$$r = \frac{V}{P}. \tag{9·6·1}$$

Hence, if no other method of rotation is used, the spin between launch and burnt is, by (4·4·4),

$$r = \frac{V_0}{P}\left(\frac{C_0}{C}\right)^{1-\nu} e^{-\tau(s_0, s)}. \tag{9·6·2}$$

If the fins are small the spin will only decrease very slowly.

It will often be convenient, however, to offset the fins at an angle Δ_F chosen so that the fins slide smoothly along the rails and remain in contact with them all along one surface. If this is done then

$$\tan \Delta_F = \frac{a_F}{P}, \tag{9·6·3}$$

where a_F is the radius to the centre of pressure of each fin (see § 3·53). If P is not too small $\Delta_F = a_F/P$ approximately, and then

$$r = \frac{1}{CP}\left\{\Gamma_F a_F \int_{t_0}^{t} V^2 \left(\frac{C'}{C}\right)^{-\nu} e^{-\tau(s', s)}\, dt' + C_0 V_0 \left(\frac{C_0}{C}\right)^{-\nu} e^{-\tau(s_0, s)}\right\}. \tag{9·6·4}$$

The same approximations that were made in § 9·5 may be applied to the right-hand side of (9·6·4) where legitimate.

9·7. Spin imparted by a combination of jet and fin rotation

Since the equation satisfied by the spin is linear, the total spin due to a number of different methods will be equal to the sum of the spins due to the separate methods, and may therefore be obtained by combining the formulae of §§ 9·4 to 9·6.

There are, however, a few points which must be borne in mind when more than one method of rotation is used.

† I.e. $2\pi P$ is the engineers' pitch.

The first point is that of the proper relation of the nozzle-inclination angle Δ_e to the fin angle Δ_F. Since the relative wind at the centre of pressure makes an angle of approximately $\tan^{-1}(a_F r/V)$ with the axis of the rocket, the fins will act as a brake on the spin unless

$$\tan\Delta_F \geqslant \frac{a_F r}{V}. \tag{9·7·1}$$

If the spin is produced by means of the jet couple G_R and the aerodynamic couple Γ_R alone, the total spin will be of the form

$$r = r_J \Delta_e + r_F \Delta_F$$

when Δ_e and Δ_F are small. We must then have

$$\frac{\Delta_F}{\Delta_e} \geqslant \frac{a_F r_J}{V - a_F r_F}.$$

It is in the early stages of flight that it is most important that the spin should be as large as possible. Now $r_F \Delta_F$ will usually be small in comparison with $r_J \Delta_e$ at launch unless Δ_e is very small. Also, by (9·42·7) and (9·43·3),

$$r_J = \frac{m_{00} a_e}{C_{00}} V,$$

approximately, in the vicinity of launch. Hence (9·7·1) becomes

$$\frac{\Delta_F}{\Delta_e} \geqslant \frac{m_{00} a_e a_F}{C_{00}}. \tag{9·7·2}$$

The second point concerns the projection of jet-rotated projectiles from spiral projectors. If the nozzle inclination is too great the fins (or other bearing surfaces) may cease contact with their corresponding rails and may foul the neighbouring rails or the supporting brackets. This will not occur if

$$\tan\Delta_e \leqslant \frac{C_{00}}{m_{00} a_e P}. \tag{9·7·3}$$

It may, of course, be advantageous to diminish the torque on the projector by choosing P and Δ_e so that (9·7·3) is satisfied by only a very narrow margin.

9·8. Parameters connected with asymmetries of design and functioning

For the unrotated rocket there are two main 'tolerances' α_N, α_R, and for the rotated rocket there are six, which cause the projectile to deviate from the trajectory of a perfect rocket, and so produce a dispersion. Five of these tolerances, namely, the angles

$$\alpha_G, \quad \alpha_L, \quad \alpha_M, \quad \alpha_N, \quad \alpha_R$$

(and their associated angles of orientation ϕ_G, ϕ_L, ϕ_M, ϕ_N, ϕ_R), depend principally upon asymmetries in the flow of the gas stream inside the rocket. These asymmetries may be due to:

(i) Malalinement, movement and distortion of the various metal components of the projectile.

(ii) Malalinement, movement, distortion and uneven composition of the propellant.

(iii) The motion of the projectile in flight, e.g. acceleration along the trajectory, and the angular velocity and acceleration about the centre of gravity.

Of the five tolerances mentioned, the last, α_R, which measures the inclination to the rocket's axis of the mean resultant gas velocity at the exit plane, is of most importance—at any rate for the unrotated rocket. The chief factor producing this so-called 'jet-malalinement' is thought to be faulty alinement of the axis of the nozzle system—the 'mechanical malalinement'. All five angles may be expected to vary during burning.

There is a sixth tolerance which has no effect on the unrotated rocket, but which may be the most important of all for highly spun rockets, namely, the angle α_C (and its associated angle of orientation ϕ_C) which measures the inclination to the rocket's axis of its longitudinal principal axis of inertia. This tolerance depends principally upon the mechanical malalinements of the various components and not upon asymmetrical gas flow caused by such malalinements. The stresses set up by the burning of the propellant may, of course, deform the projectile and so cause variations in α_C and ϕ_C. After the burning of the propellant has ceased the effect of 'inertial malalinement' continues.

In order to assign numerical values to the different tolerance angles and so predict the trajectory of the individual rocket, it is therefore necessary (i) to measure the malalinements of the various components and so calculate the 'mechanical' malalinements associated with the six angles α_P, and (ii) to calculate from these mechanical malalinements what are the actual values of the six angles α_P and their associated angles of orientation ϕ_P $(P = C, G, L, M, N, R)$.

In the case of the angle α_C the effect of the gas flow may be unimportant so that the first stage is sufficient. For the other five angles, however, the second stage cannot be ignored. We conclude by discussing how these steps are to be carried out in practice.

The first task of measuring the mechanical malalinements of the components is one which, although it presents no difficulties in theory, is by no means easy in practice. By spinning and other experimental tests it is possible to obtain a considerable amount of numerical data on the malalinements and other asymmetries of the complete projectile and its components. It is, however, difficult to make these tests either consistent or reliable, and the calculation of the mechanical malalinements from the experimental data obtained is laborious. One difficulty, which is perhaps unimportant when the dispersion of a number of projectiles is being considered, is that of choosing or defining in a consistent manner for each projectile the axis with regard to which the measurements are made. In § 3·5 the rocket's axis has been defined in a perfectly precise way, but it is clear that its determination in practice from this definition is impractical.

Because of these and other difficulties the determination of the mechanical malalinements for each individual rocket is clearly out of the question except for special experimental trials. It is, however, seldom necessary to be able to predict the precise trajectory of an individual projectile. What is of value is a knowledge of the mean point of impact of a group of projectiles and the dispersion of the group about this point. It is clear that the mean point may be assumed to be independent of the tolerances for any group of projectiles selected at random from a batch coming from one factory. The dispersion of the group will, however, depend upon the tolerances and may be expected to depend upon the dispersion of the mechanical malalinements roughly in the same way as the deviation of the individual rocket depends upon its own malalinements. For this reason it is of value to measure the standard deviation of the mechanical malalinements of a randomly selected group of projectiles coming from the same factory. This information, although laborious to obtain, is of permanent value.

The second task of calculating the malalinements relevant in the theory from their mechanical counterparts has so far proved insurmountable. It is reasonable to suppose, to take the case of the jet malalinement α_R, for example, that α_R will be proportional to the mechanical malalinement α'_R, say, but the results of firings have demonstrated conclusively that the ratio α_R/α'_R is considerably greater than unity—at any rate for malalinements of the order of those occurring in standard ammunition.† In practice, therefore, it is customary to assume some value for the factor α_R/α'_R which will make the observed dispersion agree with that calculated from the theory on the supposition of constant α_R and ϕ_R. This factor is then used to predict the dispersion of new rockets of a similar type. This process is clearly most unsatisfactory and detracts considerably from the value of the mathematical theory as a means of prediction of the effect of tolerances. It is the author's personal view, which may not find acceptance in all quarters, that the theory is not in any way discredited thereby, but that the fault lies in the lack of sufficient experimental information on the effect of mechanical malalinements and weaknesses upon the gas flow.

10. SUMMARY

10·1. General

In this section we summarize in general terms some of the main features of rocket motion as revealed by the mathematical theory. Unless it is stated to the contrary, these conclusions refer to motion during the burning period. As remarked previously, it is during this period that those disturbing factors‡ which tend to cause the projectile to deviate from the trajectory of a perfect rocket fired in ideal circumstances have their greatest effect. For this reason, the angular deviation of the projectile at the end of burning is the quantity which is of most importance for the purpose of determining the future trajectory of the projectile.

We consider the motion under the headings of gravity, wind, dispersion and stability.

10·2. Gravity

The force of gravity, acting in conjunction with the aerodynamic forces and couples, chiefly the lift moment, causes the rocket to deviate from its direction of projection. This deviation, which for the rotated rocket is not confined to the vertical plane, can be split into two parts: (i) a deviation due to the action of gravity after launch, and (ii) a deviation attributable to the conditions pertaining at the instant of launch from the projector. The first part is calculated on the assumption that the projectile is perfectly launched, i.e. that it leaves the projector with no angular deviation, yaw or cross-spin (rate of turn of axis). The second part, which may be called the tip-off correction, takes into account the actual launching conditions; it is the angular deviation due to the initial deviation, yaw and cross-spin at launch, all other disturbing forces such as gravity being ignored. Normally, by far

† It should be emphasized that the mechanical malalinement α'_R on which this ratio is based is the malalinement which is measured when the projectile is in the cold state before firing. The experimental evidence which is available is not very extensive, but, so far as it goes, confirms the view that the stresses set up during burning may alter the alinement of the nozzle axis considerably.

‡ With the exception of gravity.

the greater part of the tip-off correction (ii) is due to the cross-spin at launch, i.e. to the downwards angular velocity of the projectile as it tips off the projector. In shell ballistics (ii) is generally negligible in comparison with (i), but this is not true for rockets, since, owing to the low launching velocity, the initial cross-spin may be appreciable. One of the disadvantages of rockets in comparison with shell is that the contribution from (ii) (and, to a lesser extent, from (i)) may vary appreciably from projectile to projectile owing to variations in launching velocity.

The vertical component of the total deviation is called the 'gravity drop'. The horizontal component, which exists only for rotated projectiles, is called the drift. Whether the drift is to the right or to the left depends upon the sense of the axial spin and upon the relative magnitude of the two contributing parts. For convenience, we only consider projectiles with a right-handed spin, i.e. clockwise as viewed from the rear.† For such projectiles, the drift due to (i) is to the right or to the left according as the lift moment is destabilizing or stabilizing. After burnt the deviation, which is then identical with the lateral drift experienced by a shell, continues to build up.‡ On the other hand, the drift due to (ii) is to the left and does not build up after burnt. The magnitude of the drift (ii) is generally greater than the drift (i) at the end of burning. For this reason the lateral deviation at graze of spin-stabilized rockets is generally to the left at low Q.E.'s and to the right at high Q.E.'s, while medium-spun finned rockets generally drift slightly to the left.

10·3. Wind

The effect of wind upon the unrotated rocket is, at any rate in theory, simple. If the component of the wind from right to left is w_R, the component of the angular deviation of the rocket at burnt will be kw_R into the wind in the horizontal plane. Here k is a fixed constant depending chiefly upon the launching velocity and the magnitude of the lift moment. In a similar way the deviation in the vertical plane will be $kw_H \sin\alpha$, where w_H is the component of the wind from the front and α is the Q.E.

The behaviour of a rotating rocket in the presence of wind is not so simple. Thus the deviation due to a cross-wind is not confined to the lateral plane but may possess an appreciable component in the vertical plane. In the same way the deviation due to a head wind may possess a lateral component. Accordingly, two wind constants are required to describe the effect of wind in place of one. For finned rockets with a medium rate of spin—i.e. not sufficient to contribute to stability but sufficient to reduce dispersion—the effect of the second wind constant on the lateral deviation is small, but may be appreciable in the vertical plane. For rapidly spinning spin-stabilized rockets, however, the deviation in the plane at right angles to the wind may be the greater.

10·4. Dispersion

We consider here the effect of asymmetries of design and functioning upon the motion from the point of view of the theory and without discussing the many practical difficulties associated with the subject. Since such asymmetries are, in general, completely random from

† For left-handed spins the direction of the drift, as described in the following sentences, is reversed.

‡ The drift after burnt is not of quite the same character as the drift (i) during burning, since the former depends equally upon the lift moment and the cross velocity force, while the latter is largely independent of this force.

projectile to projectile, when a group of projectiles is fired their points of impact or burst will form a scattered pattern about the mean trajectory for a perfect projectile. Estimates of the dispersion of this scatter provide a useful measure of the accuracy of the weapon, and are, in theory, derivable from a knowledge of certain 'tolerances' and their effect upon the motion of the projectile. It appears likely that the two tolerances which can cause the greatest deviations are malalinement of the direction of the thrust—jet malalinement—and malalinement of the longitudinal principal axis of inertia of the rocket—inertial malalinement.

For the unrotated rocket the second of these has no effect. The dispersion due to the first can be reduced by increasing the launching velocity or the size of fin. The first remedy is not always practicable, and has not for some reason always proved successful when the increase in velocity has been achieved by increasing the length of travel on the projector. The second remedy has disadvantages in that it increases the sensitivity of the projectile to irregularities in the wind structure; in addition, it makes the weapon awkward to handle and to store.

By imparting rotation to the rocket the dispersion due to jet malalinement can, in theory, be reduced to insignificant proportions.† The dispersion due to inertial malalinement, on the other hand, increases as the spin is increased. Thus for most projectiles there will be an optimum spin for which the dispersion will be least. In the present state of ignorance regarding the relation between mechanical tolerances and their effect upon the gas flow, the determination of this optimum can only be accomplished by experimental means.

10·5. Stability

The problem of determining when a rocket will be stable has a number of features which distinguish it from the similar problem for shell. The true stability conditions for a rocket depend, as in the case of shell, not only upon the spin and lift moment, but also upon the other aerodynamic forces and couples such as the Magnus couple due to cross-velocity and the couple due to cross-spin. During burning, however, they depend, in addition, upon the acceleration and the damping effect of the jet. These two additional factors have, in general, a stabilizing effect upon the motion, so that a rocket may start to become unstable only after burning has ceased.

It is always possible to stabilize the projectile by providing large enough fins at the rear end, but this method often has disadvantages, e.g. too great sensitivity to gusts and awkwardness of shape. If fins are dispensed with it may be possible to achieve stability by imparting a sufficient amount of axial spin by some means. This is not, however, always so simple, since, owing to the necessity of providing a combustion chamber, the length-diameter ratio of the rocket is generally fairly high and the spin required, which depends upon this ratio, may not be realizable in practice, or may produce stresses which are great enough to deform the projectile. For this reason some compromise between 'fin' and 'spin' stability may be unavoidable.

† The assumption is made here that the jet malalinement is not appreciably affected by rotation; no information is available on this point. Some of the tolerances connected with the gas flow produce deviations which increase with the spin, so that the dispersion associated with irregularities in the gas flow cannot be reduced indefinitely by increasing the spin. It is, in fact, possible that for certain projectiles rotation may have no beneficial effect upon this dispersion.

10·6. Conclusion

In the preceding sections the mathematical theory of the motion of a rocket has been worked out in detail. By means of this theory it is possible to determine not only the trajectory during the burning period—i.e. the path of the centre of gravity—but also the yawing motion of the projectile about its centre of gravity.

The theory rests, necessarily, on certain assumptions, and it may be of value to restate those which are of most importance; the reasons for making these assumptions need not be repeated here.

In the first place, it has been assumed that the angular deviation of the trajectory remains small during the burning period. This assumption causes considerable simplification in the theory but means that it is not applicable to rockets with a long burning time.† For this reason, it would be of great interest to develop a theory independent of this assumption, but it is not to be expected that this will be possible without adding greatly to the complication of the methods employed.

Secondly, it has been assumed that the rocket is fired from the ground, or else that an appreciable part of the trajectory during the burning period is traversed at a subsonic velocity. Because of this it is justifiable to make certain assumptions regarding the form of the aerodynamic coefficients. These assumptions cannot, however, be expected to remain valid for rockets fired from aircraft where the launching velocity may be in the transonic region. It would seem desirable to extend the theory in order to cater for this type of motion, which is becoming of increasing importance. There are no inherent difficulties in making such an extension; however, until a great deal more is known about the magnitude and variation of the various aerodynamic cross-forces and couples (in particular, the Magnus couple), it seems scarcely advisable to elaborate the theory further.

Thirdly, it has been assumed that the propellant is not liquid. When this is not so, serious difficulties arise owing to the effects of rotation and acceleration. Even in the simpler case of liquid-filled shell the problem has not been solved satisfactorily, and considerable discrepancies occur between theory and practice. It is, of course, possible that the destabilizing effects of a moving liquid can be reduced or even eliminated by confining the liquid in such a way that these effects are severely restricted.

Finally, there is a great need for experimental work of a quantitative nature in every part of the theory. Of the large number of parameters, whose numerical values are required in order that the results may be applied in practice, only very few are known with any accuracy, and many are not even known to within a factor of ten. For example, to mention only one case, some of the most interesting predictions of the theory regarding stability cannot be tested, since the necessary information regarding the Magnus couple and cross-spin damping couple is not available.

APPENDIX A. TABLES OF FRESNEL FUNCTIONS

Tables are given of the Fresnel functions $A(x)$, $B(x)$ and of the subsidiary functions

$$\frac{1}{\pi x}, \quad A_1(x), \quad Z(x), \quad Z_1(x), \quad A^*(x), \quad B^*(x).$$

† It is always possible, of course, to split the trajectory up into arcs in each of which the assumption is valid.

From these the functions

$$E(u,v),\quad E^*(u,v),\quad G(u,v),\quad F(u,v),\quad H(u,v),$$

which are introduced in § 6·4, may be calculated with the help of the ordinary trigonometrical functions. The wind function $K(u,v)$ defined by (6·84·2) requires, in addition, the sine and cosine integrals. These integrals may be obtained most easily from the Work Projects Administration's tables (1940).

The definitions of the functions mentioned at the beginning of the appendix are repeated:

$$A(x) = \frac{1}{\pi\sqrt{2}}\int_0^\infty e^{-\frac{1}{2}\pi x^2 t}\frac{t^{-\frac{1}{2}}\,dt}{1+t^2} = \{\tfrac{1}{2}-S(x)\}\cos\tfrac{1}{2}\pi x^2 - \{\tfrac{1}{2}-C(x)\}\sin\tfrac{1}{2}\pi x^2, \qquad (A.\ 1)$$

$$B(x) = \frac{1}{\pi\sqrt{2}}\int_0^\infty e^{-\frac{1}{2}\pi x^2 t}\frac{t^{\frac{1}{2}}\,dt}{1+t^2} = \{\tfrac{1}{2}-S(x)\}\sin\tfrac{1}{2}\pi x^2 + \{\tfrac{1}{2}-C(x)\}\cos\tfrac{1}{2}\pi x^2, \qquad (A.\ 2)$$

$$A_1(x) = \frac{1}{\pi x} - A(x), \qquad (A.\ 3)$$

$$Z(x) = \pi\int_0^x A(u)\,du \qquad (A.\ 4)$$

$$= Z_1(x) + \log x, \qquad (A.\ 5)$$

$$A^*(x) = \int_x^\infty A(u)\frac{du}{u}, \qquad (A.\ 6)$$

$$B^*(x) = \int_x^\infty B(u)\frac{du}{u}. \qquad (A.\ 7)$$

Description of the tables

In table 1 the functions $A(x)$, $B(x)$ and $Z(x)$ are tabulated at intervals of 0·01 from $x = 0{\cdot}00$ to $x = 1{\cdot}00$ to four places of decimals. Interpolation may be carried out by using the first differences only.

In table 2 the functions $A(x)$, $A_1(x)$, $B(x)$, $Z(x)$, $Z_1(x)$ and $1/\pi x$ are tabulated at varying intervals from $x = 1{\cdot}00$ to $x = 15{\cdot}0$. Interpolation may be carried out by using the first differences for all these functions with the exception of $Z(x)$, when x exceeds 2. When x is less than two, second differences are occasionally necessary, and Bessel's central difference formula may be employed.

For the function $Z(x)$ both first and second differences are given throughout. When tables of Napierian logarithms are available an alternative method of calculating $Z(x)$ is to use the table of the slowly varying function $Z_1(x)$ in conjunction with the formula (A. 5). This method is more convenient when interpolation is necessary.

It will be observed that there is little difference between the entries for $A(x)$ and $1/\pi x$ in the latter part of the table. It was thought to be advantageous, however, to tabulate these functions separately so that the last figure might be correct in every case. For the same reason, the function $A_1(x)$ is given at each point although it can easily be evaluated from the values of $A(x)$ and $1/\pi x$ by use of (A. 3).

In tables 3 and 4 the functions $A^*(x)$ and $B^*(x)$ are tabulated at intervals of 0·1 from 0·0 to 5·0. In the region 0·0 to 1·0 it may be more convenient when interpolating to use the tables of $A^*(x)+\tfrac{1}{2}\log x$ and $B^*(x)+\tfrac{1}{2}\log x$.

TABLE 1

x	$A(x)$	ΔA	$B(x)$	ΔB	$Z(x)$	ΔZ	x	$A(x)$	ΔA	$B(x)$	ΔB	$Z(x)$	ΔZ
		−		−		+			−		−		+
0·00	0·5000	1	0·5000	99	0·0000	157	0·50	0·3992	27	0·1736	37	0·7227	125
0·01	0·4999	2	0·4901	98	0·0157	157	0·51	0·3965	27	0·1699	36	0·7352	124
0·02	0·4997	4	0·4803	96	0·0314	157	0·52	0·3938	28	0·1663	35	0·7476	123
0·03	0·4993	5	0·4707	94	0·0471	157	0·53	0·3910	27	0·1628	34	0·7599	122
0·04	0·4988	6	0·4613	93	0·0628	156	0·54	0·3883	27	0·1594	34	0·7721	122
0·05	0·4982	8	0·4520	92	0·0784	157	0·55	0·3856	27	0·1560	33	0·7843	121
0·06	0·4974	9	0·4428	90	0·0941	156	0·56	0·3829	26	0·1527	32	0·7964	119
0·07	0·4965	10	0·4338	88	0·1097	156	0·57	0·3803	27	0·1495	32	0·8083	120
0·08	0·4955	11	0·4250	87	0·1253	155	0·58	0·3776	27	0·1463	31	0·8203	118
0·09	0·4944	13	0·4163	85	0·1408	155	0·59	0·3749	26	0·1432	30	0·8321	117
0·10	0·4931	13	0·4078	84	0·1563	155	0·60	0·3723	27	0·1402	29	0·8438	117
0·11	0·4918	14	0·3994	82	0·1718	154	0·61	0·3696	26	0·1373	29	0·8555	115
0·12	0·4904	15	0·3912	81	0·1872	154	0·62	0·3670	26	0·1354	28	0·8670	115
0·13	0·4889	16	0·3831	79	0·2026	154	0·63	0·3644	26	0·1316	28	0·8785	114
0·14	0·4873	17	0·3752	78	0·2180	152	0·64	0·3618	26	0·1288	27	0·8899	114
0·15	0·4856	18	0·3674	76	0·2332	153	0·65	0·3592	25	0·1261	26	0·9013	112
0·16	0·4838	19	0·3598	75	0·2485	151	0·66	0·3567	26	0·1235	26	0·9125	112
0·17	0·4819	19	0·3523	74	0·2636	151	0·67	0·3541	25	0·1209	25	0·9237	111
0·18	0·4800	20	0·3449	72	0·2787	151	0·68	0·3516	25	0·1184	25	0·9348	110
0·19	0·4780	20	0·3377	71	0·2938	150	0·69	0·3491	25	0·1159	24	0·9458	109
0·20	0·4760	21	0·3306	69	0·3088	149	0·70	0·3466	25	0·1135	23	0·9567	108
0·21	0·4739	22	0·3237	68	0·3237	149	0·71	0·3441	25	0·1112	23	0·9675	108
0·22	0·4717	22	0·3169	67	0·3386	147	0·72	0·3416	25	0·1089	23	0·9783	107
0·23	0·4695	23	0·3102	66	0·3533	148	0·73	0·3391	24	0·1066	22	0·9890	106
0·24	0·4672	23	0·3036	64	0·3681	146	0·74	0·3367	24	0·1044	21	0·9996	106
0·25	0·4649	23	0·2972	62	0·3827	146	0·75	0·3343	24	0·1023	21	1·0102	104
0·26	0·4626	24	0·2910	62	0·3973	145	0·76	0·3319	24	0·1002	21	1·0206	104
0·27	0·4602	25	0·2848	60	0·4118	144	0·77	0·3295	24	0·0981	20	1·0310	103
0·28	0·4577	24	0·2788	59	0·4262	143	0·78	0·3271	23	0·0961	20	1·0413	103
0·29	0·4553	25	0·2729	58	0·4405	143	0·79	0·3248	23	0·0941	19	1·0516	101
0·30	0·4528	26	0·2671	57	0·4548	142	0·80	0·3225	23	0·0922	18	1·0617	101
0·31	0·4502	25	0·2614	56	0·4690	141	0·81	0·3202	23	0·0904	19	1·0718	100
0·32	0·4477	26	0·2558	54	0·4831	140	0·82	0·3179	23	0·0885	18	1·0818	100
0·33	0·4451	26	0·2504	53	0·4971	139	0·83	0·3156	23	0·0867	17	1·0918	99
0·34	0·4425	26	0·2451	53	0·5110	139	0·84	0·3133	22	0·0850	17	1·1017	98
0·35	0·4399	27	0·2398	51	0·5249	138	0·85	0·3111	22	0·0833	17	1·1115	97
0·36	0·4372	26	0·2347	50	0·5387	137	0·86	0·3089	22	0·0816	16	1·1212	97
0·37	0·4346	27	0·2297	49	0·5524	136	0·87	0·3067	22	0·0800	16	1·1309	96
0·38	0·4319	27	0·2248	48	0·5660	135	0·88	0·3045	21	0·0784	16	1·1405	95
0·39	0·4292	27	0·2200	47	0·5795	134	0·89	0·3024	22	0·0768	15	1·1500	95
0·40	0·4265	27	0·2153	45	0·5929	134	0·90	0·3002	21	0·0753	15	1·1595	94
0·41	0·4238	27	0·2108	45	0·6063	133	0·91	0·2981	21	0·0738	15	1·1689	93
0·42	0·4211	28	0·2063	44	0·6196	132	0·92	0·2960	21	0·0723	14	1·1782	93
0·43	0·4183	27	0·2019	43	0·6328	131	0·93	0·2939	20	0·0709	14	1·1875	92
0·44	0·4156	27	0·1976	42	0·6459	130	0·94	0·2919	21	0·0695	14	1·1967	91
0·45	0·4129	28	0·1934	42	0·6589	129	0·95	0·2898	20	0·0681	13	1·2058	91
0·46	0·4101	27	0·1892	40	0·6718	128	0·96	0·2878	20	0·0668	13	1·2149	90
0·47	0·4074	27	0·1852	39	0·6846	128	0·97	0·2858	20	0·0655	13	1·2239	90
0·48	0·4047	28	0·1813	39	0·6974	127	0·98	0·2838	20	0·0642	13	1·3329	88
0·49	0·4019	27	0·1774	38	0·7101	126	0·99	0·2818	19	0·0629	12	1·2417	88
0·50	0·3992		0·1736		0·7227		1·00	0·2799		0·0617		1·2505	

TABLE 2

x	$A(x)$	ΔA	$B(x)$	ΔB	$A_1(x)$	ΔA_1	$Z(x)$	ΔZ	$\Delta^2 Z$	$Z_1(x)$	ΔZ_1	$\frac{1}{\pi x}$	Δ
		—		—		—		+	—		—		—
1·00	0·2799	95	0·0617	56	0·0384	57	1·2505	432	15	1·2505	56	0·3183	151
1·05	0·2704	90	0·0561	51	0·0327	47	1·2937	418	14	1·2449	47	0·3032	138
1·10	0·2614	86	0·0510	46	0·0280	40	1·3355	404	14	1·2402	40	0·2894	126
1·15	0·2528	82	0·0464	41	0·0240	33	1·3759	391	13	1·2362	35	0·2768	115
1·20	0·2446	78	0·0423	37	0·0207	29	1·4150	378	13	1·2327	30	0·2653	107
1·25	0·2368	73	0·0386	33	0·0178	24	1·4528	366	12	1·2297	26	0·2546	97
1·30	0·2295	70	0·0353	30	0·0154	21	1·4894	355	11	1·2271	23	0·2449	91
1·35	0·2225	67	0·0323	27	0·0133	17	1·5249	344	11	1·2248	20	0·2358	84
1·40	0·2158	64	0·0296	24	0·0116	15	1·5593	334	10	1·2228	17	0·2274	79
1·45	0·2094	60	0·0272	22	0·0101	13	1·5927	324	10	1·2211	15	0·2195	73
1·50	0·2034		0·0250		0·0088		1·6251		10	1·2196		0·2122	
1·5	0·2034	112	0·0250	37	0·0088	21	1·6251	621	37	1·2196	24	0·2122	133
1·6	0·1922	102	0·0213	31	0·0067	15	1·6872	588	33	1·2172	19	0·1989	117
1·7	0·1820	93	0·0182	26	0·0052	11	1·7460	557	31	1·2153	14	0·1872	104
1·8	0·1727	84	0·0156	21	0·0041	9	1·8017	529	28	1·2139	11	0·1768	93
1·9	0·1643	77	0·0135	18	0·0032	6	1·8546	504	25	1·2128	9	0·1675	83
2·0	0·1566	71	0·0117	14	0·0026	5	1·9050	481	23	1·2119	7	0·1592	76
2·1	0·1495	65	0·0103	13	0·0021	4	1·9531	459	22	1·2112	6	0·1516	69
2·2	0·1430	60	0·0090	11	0·0017	3	1·9990	440	19	1·2106	5	0·1447	63
2·3	0·1370	55	0·0079	9	0·0014	3	2·0430	422	18	1·2101	4	0·1384	58
2·4	0·1315	51	0·0070	7	0·0011	2	2·0852	405	17	1·2097	3	0·1326	53
2·5	0·1264	47	0·0063	7	0·0009	1	2·1257	390	15	1·2094	2	0·1273	49
2·6	0·1217	44	0·0056	6	0·0008	2	2·1647	375	15	1·2092	3	0·1224	45
2·7	0·1173	42	0·0050	5	0·0006	1	2·2022	362	13	1·2089	2	0·1179	42
2·8	0·1131	38	0·0045	4	0·0005	0	2·2384	349	13	1·2087	1	0·1137	39
2·9	0·1093	36	0·0041	4	0·0005	1	2·2733	338	11	1·2086	1	0·1098	37
3·0	0·1057	33	0·0037	4	0·0004	1	2·3071	327	11	1·2085	1	0·1061	34
3·1	0·1024	32	0·0033	3	0·0003	0	2·3398	316	11	1·2084	1	0·1027	32
3·2	0·0992	30	0·0030	2	0·0003	1	2·3714	307	9	1·2083	1	0·0995	30
3·3	0·0962	28	0·0028	3	0·0002	0	2·4021	298	9	1·2082	1	0·0965	29
3·4	0·0934	26	0·0025	2	0·0002	0	2·4319	289	9	1·2081	1	0·0936	27
3·5	0·0908	25	0·0023	1	0·0002	0	2·4608	281	8	1·2080	0	0·0909	25
3·6	0·0883	24	0·0022	2	0·0002	1	2·4889	274	7	1·2080	1	0·0884	24
3·7	0·0859	23	0·0020	2	0·0001	0	2·5163	266	8	1·2079	0	0·0860	22
3·8	0·0836	21	0·0018	1	0·0001	0	2·5429	259	7	1·2079	0	0·0838	22
3·9	0·0815	20	0·0017	1	0·0001	0	2·5688	253	6	1·2079	1	0·0816	20
4·0	0·0795	19	0·0016	1	0·0001	0	2·5941	247	6	1·2078	0	0·0796	20
4·1	0·0776	19	0·0015	1	0·0001	0	2·6188	241	6	1·2078	0	0·0776	18
4·2	0·0757	17	0·0014	1	0·0001	0	2·6429	235	6	1·2078	0	0·0758	18
4·3	0·0740	17	0·0013	1	0·0001	0	2·6664	230	5	1·2078	1	0·0740	17
4·4	0·0723	16	0·0012	1	0·0001	0	2·6894	224	6	1·2077	0	0·0723	16
4·5	0·0707	15	0·0011	1	0·0001	1	2·7118	220	4	1·2077	0	0·0707	15
4·6	0·0692	15	0·0010	0	0·0000	0	2·7338	215	5	1·2077	0	0·0692	15
4·7	0·0677	14	0·0010	1	0·0000	0	2·7553	210	5	1·2077	0	0·0677	14
4·8	0·0663	14	0·0009	0	0·0000	0	2·7763	206	4	1·2077	0	0·0663	13
4·9	0·0649	13	0·0009	1	0·0000	0	2·7969	202	4	1·2077	0	0·0650	13
5·0	0·0636		0·0008		0·0000		2·8171		4	1·2077		0·0637	

TABLE 2 (*continued*)

x	$A(x)$	ΔA	$B(x)$	$Z(x)$	ΔZ	$\Delta^2 Z$	$Z_1(x)$	$\dfrac{1}{\pi x}$	Δ
		−			+	−			−
5·0	0·0636	12	0·0008	2·8171	198	4	1·2077	0·0637	13
5·1	0·0624	12	0·0008	2·8369	194	4	1·2077	0·0624	12
5·2	0·0612	12	0·0007	2·8563	190	4	1·2077	0·0612	11
5·3	0·0600	11	0·0007	2·8753	187	3	1·2076	0·0601	12
5·4	0·0589	10	0·0006	2·8940	184	3	1·2076	0·0589	10
5·5	0·0579	11	0·0006	2·9124	180	4	1·2076	0·0579	11
5·6	0·0568	10	0·0006	2·9304	177	3	1·2076	0·0568	10
5·7	0·0558	9	0·0005	2·9481	174	3	1·2076	0·0558	9
5·8	0·0549	10	0·0005	2·9655	171	3	1·2076	0·0549	9
5·9	0·0539	9	0·0005	2·9826	168	3	1·2076	0·0540	9
6·0	0·0530	8	0·0005	2·9994	165	3	1·2076	0·0531	9
6·1	0·0522	9	0·0004	3·0159	162	3	1·2076	0·0522	9
6·2	0·0513	8	0·0004	3·0321	160	2	1·2076	0·0513	8
6·3	0·0505	8	0·0004	3·0481	158	2	1·2076	0·0505	8
6·4	0·0497	7	0·0004	3·0639	155	3	1·2076	0·0497	7
6·5	0·0490	8	0·0004	3·0794	153	2	1·2076	0·0490	8
6·6	0·0482	7	0·0004	3·0947	150	3	1·2076	0·0482	7
6·7	0·0475	7	0·0003	3·1097	148	2	1·2076	0·0475	7
6·8	0·0468	7	0·0003	3·1245	146	2	1·2076	0·0468	7
6·9	0·0461	6	0·0003	3·1391	144	2	1·2076	0·0461	6
7·0	0·0455		0·0003	3·1535		2	1·2076	0·0455	
7·0	0·0455	13	0·0003	3·1535	282	8	1·2076	0·0455	13
7·2	0·0442	12	0·0003	3·1817	274	8	1·2076	0·0442	12
7·4	0·0430	11	0·0003	3·2091	266	8	1·2076	0·0430	11
7·6	0·0419	11	0·0002	3·2357	260	6	1·2076	0·0419	11
7·8	0·0408	10	0·0002	3·2617	253	7	1·2076	0·0408	10
8·0	0·0398	10	0·0002	3·2870	247	6	1·2076	0·0398	10
8·2	0·0388	9	0·0002	3·3117	241	6	1·2076	0·0388	9
8·4	0·0379	9	0·0002	3·3358	235	6	1·2076	0·0379	9
8·6	0·0370	8	0·0002	3·3593	230	5	1·2076	0·0370	8
8·8	0·0362	8	0·0001	3·3823	225	5	1·2076	0·0362	8
9·0	0·0354	8	0·0001	3·4048	220	5	1·2076	0·0354	8
9·2	0·0346	7	0·0001	3·4268	215	5	1·2076	0·0346	7
9·4	0·0339	7	0·0001	3·4483	210	5	1·2076	0·0339	7
9·6	0·0332	7	0·0001	3·4693	206	4	1·2076	0·0332	7
9·8	0·0325	7	0·0001	3·4899	202	4	1·2076	0·0325	7
10·0	0·0318		0·0001	3·5101		4	1·2076	0·0318	
10·0	0·0318	15	0·0001	3·5101	488	25	1·2076	0·0318	15
10·5	0·0303	14	0·0001	3·5589	465	23	1·2076	0·0303	14
11·0	0·0289	12	0·0001	3·6054	445	20	1·2076	0·0289	12
11·5	0·0277	12	0·0000	3·6499	426	19	1·2076	0·0277	12
12·0	0·0265	10	0·0000	3·6925	408	18	1·2076	0·0265	10
12·5	0·0255	10	0·0000	3·7333	392	16	1·2075	0·0255	10
13·0	0·0245	9	0·0000	3·7725	377	15	1·2075	0·0245	9
13·5	0·0236	9	0·0000	3·8102	364	13	1·2075	0·0236	9
14·0	0·0227	7	0·0000	3·8466	351	13	1·2075	0·0227	7
14·5	0·0220	8	0·0000	3·8817	339	12	1·2075	0·0220	8
15·0	0·0212		0·0000	3·9156		11	1·2075	0·0212	

For x greater than 15, $A(x) = \dfrac{1}{\pi x}$, $B(x) = 0$, $Z(x) = 1{\cdot}2075 + \log x$, correct to four figures.

$$Z_1(\infty) = 1{\cdot}207546366.$$

TABLE 3

x	$A^*(x)$	ΔA^*	$\Delta^2 A^*$	$B^*(x)$	ΔB^*	$\Delta^2 B^*$	$A^*(x)+\frac{1}{2}\log x$	Δ	Δ^2	$B^*(x)+\frac{1}{2}\log x$	Δ	Δ^2
0·0	∞	—		∞			0·1355	36		−0·6499	961	—
0·1	1·2904	3370		0·5975	—		0·1391	96	60	−0·5538	884	77
					2582							
0·2	0·9534	1888	1482	0·3393	1216	1366	0·1487	139	43	−0·4654	811	73
0·3	0·7646	1267	621	0·2177	695	521	0·1626	172	33	−0·3843	744	67
0·4	0·6379	922	345	0·1482	434	261	0·1798	193	21	−0·3099	681	63
0·5	0·5457	704	218	0·1048	286	148	0·1991	208	15	−0·2418	626	56
0·6	0·4753	554	150	0·0762	197	89	0·2199	216	8	−0·1792	576	50
0·7	0·4199	447	107	0·0567	137	60	0·2415	221	5	−0·1216	530	46
0·8	0·3752	367	80	0·0430	99	38	0·2636	222	1	−0·0686	491	39
0·9	0·3385	306	61	0·0331	72	27	0·2858	221	−1	−0·0195	454	37
1·0	0·3079	258	48	0·0259	53	19	0·3079			+0·0259		

TABLE 4

x	$A^*(x)$	ΔA^*	$\Delta^2 A^*$	$B^*(x)$	ΔB^*	$\Delta^2 B^*$
		—			—	
1·1	0·2821	220	38	0·0206	41	12
1·2	0·2601	189	31	0·0165	31	10
1·3	0·2412	165	24	0·0134	24	7
1·4	0·2247	145	20	0·0110	19	5
1·5	0·2102	128	17	0·0091	14	5
1·6	0·1974	113	15	0·0077	12	2
1·7	0·1861	102	11	0·0065	10	2
1·8	0·1759	91	11	0·0055	8	2
1·9	0·1668	82	9	0·0047	6	2
2·0	0·1586	75	7	0·0041	6	
2·1	0·1511	68	7	0·0035	4	
2·2	0·1443	62	6	0·0031	4	
2·3	0·1381	57	5	0·0027	3	
2·4	0·1324	53	4	0·0024	3	
2·5	0·1271	48	5	0·0021	2	
2·6	0·1223	45	3	0·0019	2	
2·7	0·1178	42	3	0·0017	2	
2·8	0·1136	39	3	0·0015	1	
2·9	0·1097	37	2	0·0014	2	
3·0	0·1060	34	3	0·0012	1	

x	$A^*(x)$	ΔA^*	$B^*(x)$
		—	
3·1	0·1026	32	0·0011
3·2	0·0994	30	0·0010
3·3	0·0964	28	0·0009
3·4	0·0936	27	0·0009
3·5	0·0909	25	0·0008
3·6	0·0884	24	0·0007
3·7	0·0860	23	0·0007
3·8	0·0837	21	0·0006
3·9	0·0816	20	0·0006
4·0	0·0796	20	0·0005
4·1	0·0776	18	0·0005
4·2	0·0758	18	0·0005
4·3	0·0740	17	0·0004
4·4	0·0723	16	0·0004
4·5	0·0707	15	0·0004
4·6	0·0692	15	0·0003
4·7	0·0677	14	0·0003
4·8	0·0663	13	0·0003
4·9	0·0650	13	0·0003
5·0	0·0637		0·0003

For $x > 5{\cdot}0$, $A^*(x) = \dfrac{1}{\pi x}$, $B^*(x) = \frac{1}{3}B(x)$ correct to four figures.

Method of calculation of the tables

The formulae (6·4·5) provide a direct method of calculating $A(x)$ and $B(x)$ for any value of x. This method was not employed, however, in the construction of the tables which was done as follows.

Table 1. Values correct to seven places of decimals of $C(x)$ and $S(x)$ for $x = 0, 0{\cdot}1, 0{\cdot}2, \ldots, 1{\cdot}0$, which had been calculated from the series expansions, were used to obtain $A(x)$ and $B(x)$ at each of these eleven points with the help of (6·4·4). From these the values of $A(x)$ and $B(x)$ at the ten intermediate points $x = 0{\cdot}05, 0{\cdot}15, \ldots, 0{\cdot}95$ were derived by means of the Gregory-Newton interpolation formula by using differences up to the sixth order so as to obtain accuracy to five places of decimals. The differences between the values at the 21 points $x = 0{\cdot}00, 0{\cdot}05, 0{\cdot}10, \ldots, 1{\cdot}00$ were then sufficiently smooth for the values of $A(x)$ and $B(x)$ to be obtained correct to four figures at intervals of 0·01 by use of Bessel's formula.

Table 2. (*a*) $1{\cdot}05 \leqslant x \leqslant 1{\cdot}9$. The values of $A(x)$ and $B(x)$ in this range have been taken from the tables given by Lash Miller and Gordon (1931), and the last figure may not be true in some cases. The function $A_1(x)$, $Z(x)$ and $Z_1(x)$ have been calculated from them, so that the same doubt attaches to the last figure of $A_1(x)$, and, to a lesser extent, to the last figure of $Z(x)$ and $Z_1(x)$.

(*b*) $2{\cdot}0 \leqslant x \leqslant 15{\cdot}0$. All the values in this range have been recalculated. For the functions $A(x)$, $A_1(x)$ and $B(x)$, this was done by means of the asymptotic formulae

$$A(x) = \sigma_0 - \sigma_2 + \sigma_4 - \dots + (-1)^n(\sigma_{2n} - R_{2n}),$$
$$A_1(x) = \sigma_2 - \sigma_4 + \dots - (-1)^n(\sigma_{2n} - R_{2n}),$$
$$B(x) = \sigma_1 - \sigma_3 + \sigma_3 - \dots + (-1)^n(\sigma_{2n+1} - R_{2n+1}),$$

where
$$\sigma_n = \frac{1}{\pi\sqrt{2}}\Gamma(n+\tfrac{1}{2})(\tfrac{1}{2}\pi x^2)^{-n-\frac{1}{2}} \tag{A. 8}$$
and the 'remainder' R_n is given by
$$R_n = \frac{1}{\pi\sqrt{2}}\int_0^\infty e^{-\frac{1}{2}\pi x^2 t}\frac{t^{n+\frac{3}{2}}}{1+t^2}\,dt. \tag{A. 9}$$

For example $\sigma_0 = \dfrac{1}{\pi x}$, $\sigma_1 = \dfrac{1}{\pi^2 x^3}$, $\sigma_2 = \dfrac{3}{\pi^3 x^5}, \dots,$ $\sigma_n = \dfrac{1\,.\,3\,.\,\dots\,.\,(2n-1)}{\pi^{n+1}\,x^{2n+1}}$.

It is clear from (A. 8, 9) that the remainder satisfies the inequalities
$$0 < R_n < \sigma_n, \quad 0 < R_n < \sigma_{n+2}. \tag{A. 10}$$
These inequalities, however, are insufficient to determine $A(x)$ and $B(x)$ correctly to four figures of decimals when the argument lies between 2 and 3.

Accordingly, use was made of the following artifice (cf. Rankin 1945*b*). For positive t,
$$\frac{1}{t} - \frac{1}{4}\left(1+\frac{1}{t^2}\right) \leqslant \frac{1}{1+t^2} \leqslant \frac{1}{2t},$$
and it follows from (A. 8, 9) that
$$\sigma_{n+1} - \tfrac{1}{4}(\sigma_n + \sigma_{n+2}) \leqslant R_n \leqslant \tfrac{1}{2}\sigma_{n+1}. \tag{A. 11}$$
This inequality usually confines the error to narrower bounds than are obtained from (A. 10). By this means it is possible to use the asymptotic formulae to obtain the values of $A(x)$ and $B(x)$ for values of x from 2 onwards.

The functions $Z(x)$ and $Z_1(x)$ were calculated from the asymptotic formula
$$Z(x) = \log x + Z_1(x)$$
$$= \log x + \tfrac{1}{2}(\gamma + \log 2\pi) + \frac{1\,.\,3}{4\pi^2 x^4} - \frac{1\,.\,3\,.\,5\,.\,7}{8\pi^4 x^8} + \frac{1\,.\,3\,.\,5\,.\,7\,.\,9\,.\,11}{12\pi^6 x^{12}} - \dots.$$

By means of an artifice similar to that described above, this formula can be applied to give results accurate to four figures of decimals for $x \geqslant 2$. The γ appearing in the formula is Euler's constant, and
$$\tfrac{1}{2}(\gamma + \log 2\pi) = 1{\cdot}20754637,$$
approximately.

Tables 3 *and* 4. (*a*) $0 \leqslant x \leqslant 1{\cdot}0$. The values of $A^*(x)$ and $B^*(x)$ in this range have been calculated from the formulae (6·82·4, 5).

(*b*) $1{\cdot}1 \leqslant x \leqslant 1{\cdot}9$. The values in this range have been calculated by numerical integration from the values of $A(x)$ and $B(x)$.

(*c*) $2{\cdot}0 \leqslant x \leqslant 5{\cdot}0$. For these values of the argument the asymptotic formulae (6·82·6, 7) have been used together with the artifice described above.

Other tables. There are two definitions in use of the Fresnel integrals C and S. (For tables of the functions defined by (6·4·2) see Edwards (1922, p. 332), Preston (1928, p. 296) or Jahnke & Emde (1938, p. 34).) According to the alternative definition

$$C(z) = \int_0^z \frac{\cos t}{(2\pi t)^{\frac{1}{2}}} dt, \quad S(z) = \int_0^z \frac{\sin t}{(2\pi t)^{\frac{1}{2}}} dt.$$

These definitions agree with the formulae (6·4·1, 2) when $z = \frac{1}{2}\pi x^2$. Tables, to four places of decimals, may be found in Jahnke & Emde (1938, p. 35), and, to a greater number of places, in Watson (1922) and in the *British Association Report* (1926).

For the functions $$A(x) = \frac{1}{\surd\pi}(K+H), \quad B(x) = \frac{1}{\surd\pi}(K-H)$$

see Lash Miller & Gordon (1931, p. 2873). Part of the table given there is included in Rankin (1943 *b*).

The tables of $C(x)$ and $S(x)$ in Edwards (1922), Preston (1928) and Jahnke & Emde (1938) are to four places of decimals at intervals of 0·1. A list of errors occurring in the latter is given in *Mathematical tables and aids to computation* (1945, pp. 395, 398).

The table of $A(x)$ and $B(x)$ given by Lash Miller & Gordon (1931) proceeds in steps of 0·05 from 0 to 1·50, in steps of 0·1 from 1·5 to 8·5, and in steps of 0·5 from 8·5 to 15·0. The values given are to four places of decimals, and the last figure is not always true.

The functions rr (x) and rj (x) tabulated by Rosser, Newton & Gross (1947) are connected with the functions $A(x)$ and $B(x)$ by the relations

$$A(x) = (2\pi)^{-\frac{1}{2}} \operatorname{rr}(\tfrac{1}{2}\pi x^2), \quad B(x) = (2\pi)^{-\frac{1}{2}} \operatorname{rj}(\tfrac{1}{2}\pi x^2).$$

APPENDIX B. NUMERICAL EXAMPLES AND EXPLANATION OF FIGURES 6 TO 15

The formulae of § 6 have been applied to various spin- and fin-stabilized rockets rotated by means of inclined nozzles. In order to present a full picture of the behaviour of such a rocket during burning, detailed calculations of the angular deviation, yaw, etc., have been made for a number of instants during burning, and curves have been drawn from the figures obtained. These calculations have, for convenience, been based on a fictitious but representative rocket whose characteristics have been chosen in order to make the numerical work as simple as possible. The relevant parameters chosen for this rocket are:

Time of burning	$t_1 = 1$ sec.
Launching velocity	$V_0 = 150$ ft./sec.
All burnt velocity	$V_1 = 1500$ ft./sec.
Constant acceleration	$f = 1500$ ft./sec.^{2}
Spin-velocity ratio	$\beta = 0{\cdot}025$
	$\gamma = 1$
Destabilizing moment coefficient	$\nu^2 = 2{\cdot}25 \times 10^{-4}$ ft.$^{-2}$

From these figures we derive

$$\lambda_1 = 0{\cdot}045, \quad \lambda_2 = 0{\cdot}005, \quad p = 0{\cdot}02\,\text{ft.}^{-1}.$$

The stability factor is 2·778. Values of the mass, moments of inertia, length, number of nozzles, nozzle inclinations, etc., are not given, since they do not affect the values of the deviation and yaw except as constant multiplying factors. Any reasonable values which are

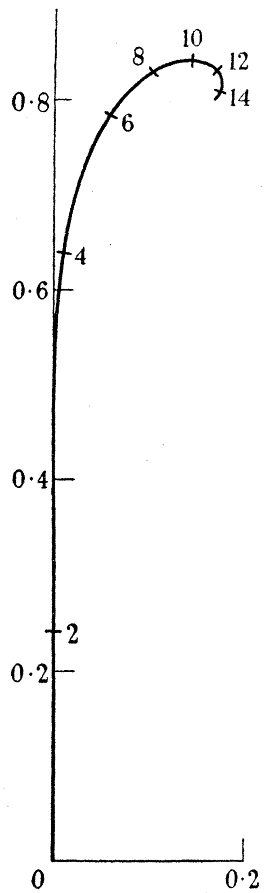

FIGURE 6. Angular deviation due to an initial yaw of 1°. (Units = degrees.)

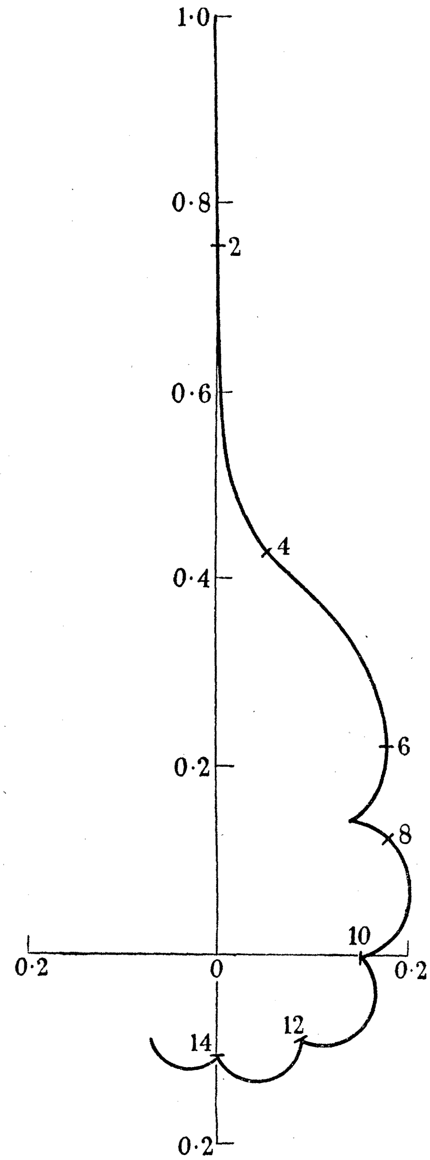

FIGURE 7. Yaw due to an initial yaw of 1°. (Units = degrees.)

Figures 6, 7. *Initial yaw.* Perfect launch except for an initial yaw of 1° has been assumed. This means that the rocket is launched with its nose pointing 1° *above* the trajectory ($\Xi_0 = \zeta_0 = 1°$). The deviations and yaws due to initial yaws of other magnitudes and orientations can be deduced by magnifying or diminishing in the required ratio, and orientation through the necessary angle.

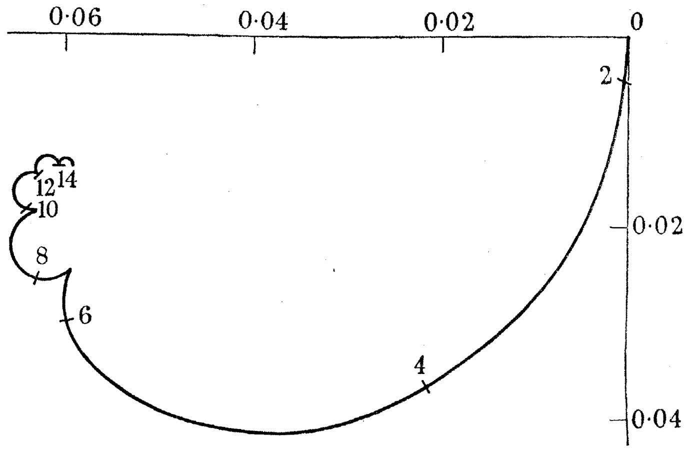

FIGURE 8. Angular deviation due to an initial rate of turn of 1° per sec. (Units = degrees.)

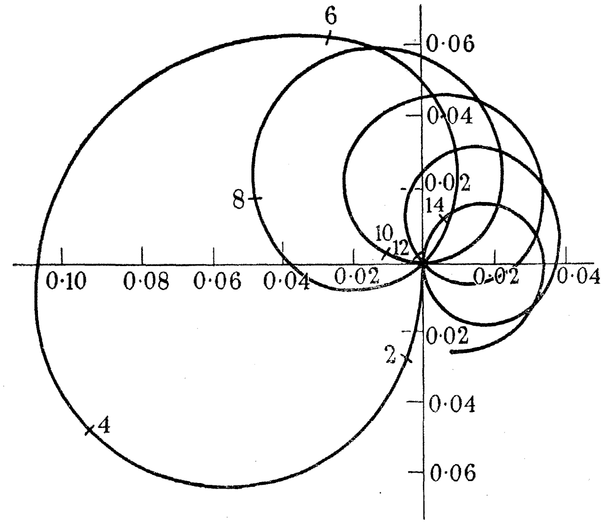

FIGURE 9. Yaw due to an initial rate of turn of 1° per sec. (Units = degrees.)

Figures 8, 9. *Initial rate of turn* (*cross-spin*). An initial rate of turn of the axis of 1°/sec. downwards in the vertical plane has been taken (i.e. $\zeta_{01} = 1°$/sec.). It will be observed that there is a considerable deviation to the left. The deviations and yaws corresponding to other rates of turn can be found as described above.

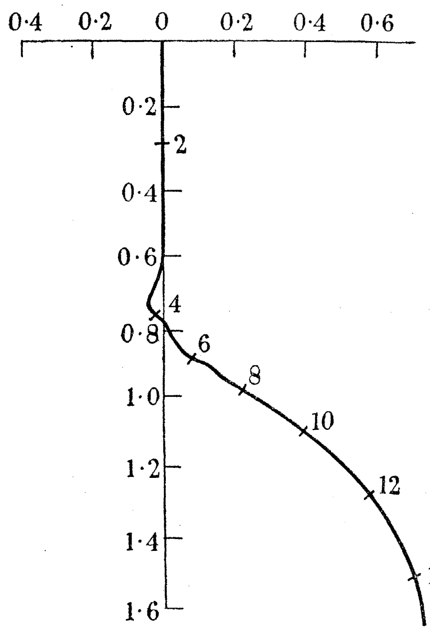

FIGURE 10. Angular deviation due to gravity.

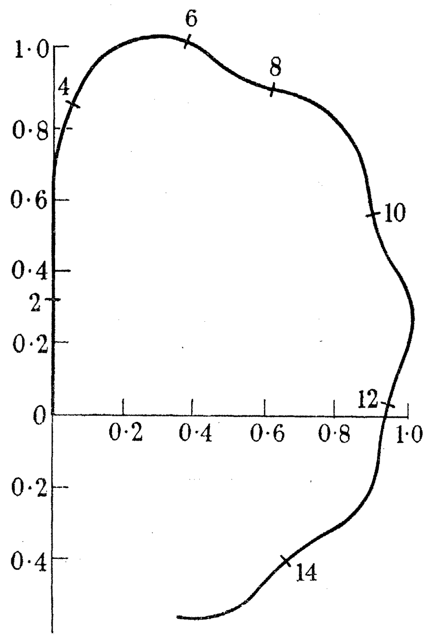

FIGURE 11. Yaw due to gravity.

(Perfect launch is assumed. Q.E. 0°. Multiply by cos Q.E. for other Q.E.'s.)
(Units = degrees.)

Figures 10, 11. *Gravity*. The figure shows the angular deviation and yaw due to gravity for zero Q.E. Perfect launch is assumed so that there is a drift to the right. When the tip-off components are known their effect can be determined from figures 6 to 9. It is clear from figure 8 that the final deviation at burnt will be to the left instead of to the right if the initial tip-off angular velocity is appreciable. The deviation and yaw at a Q.E. α may be found from the figures by multiplication by cos α.

consistent with the figures given may be chosen. The following values are quoted as an indication of the type of weapon to which the above figures could apply:

Length		$= 3\frac{1}{2}$ ft. approx.	Calibre		= 7 in. approx.
Mass (initial)	m_{00}	= 100 lb.	Mass (final)	m_1	= 78 lb.
Transverse M.I.	A_{00}	= 100 lb.ft.^{2}	Axial M.I.	C_{00}	= 5 lb.ft.^{2}
Nozzle inclination		= 15°	Nozzle ring radius		= 2·24 in.
Projector length		$= 7\frac{1}{2}$ ft.	Gas velocity	W	= 6250 ft./sec.

$$ml/A = 1{\cdot}8 \text{ ft.}^{-1} \text{ approx.}$$

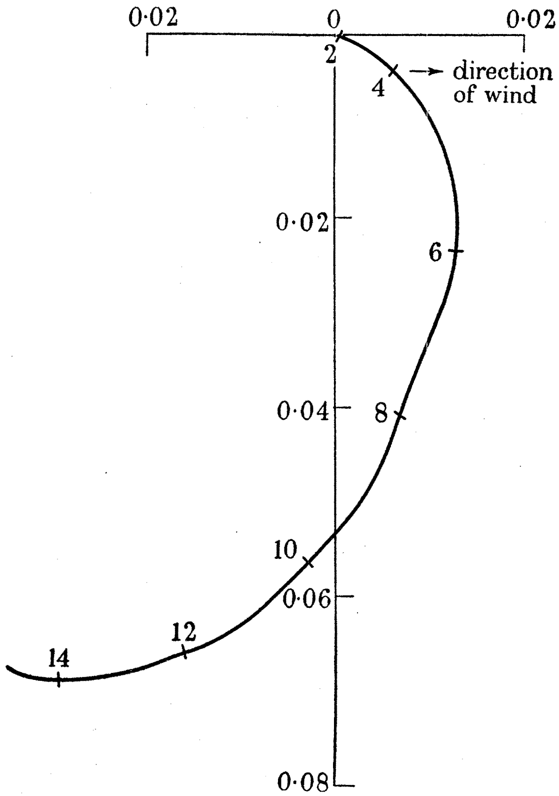

FIGURE 12. Angular deviation due to a constant cross-wind.

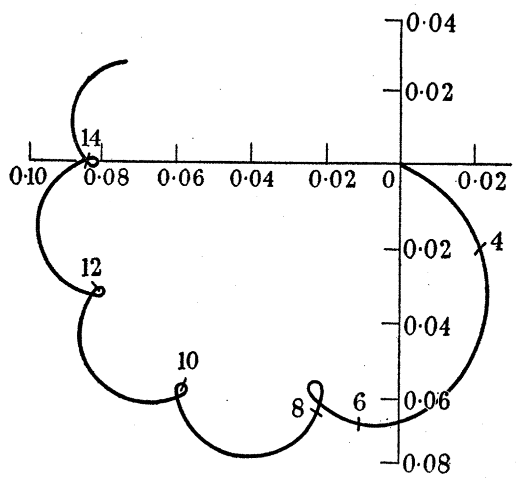

FIGURE 13. Yaw due to a constant cross-wind.

(Wind speed 1 ft./sec. from left to right.)
(Units = degrees.)

Figures 12, 13. *Wind.* The figures show the angular deviation and yaw due to a cross-wind blowing from *left to right* across the line of fire, and of speed 1 ft./sec. The deviations and yaws corresponding to other magnitudes and directions of the wind can be found from the figure by a suitable magnification and orientation (see § 6·83).

Calculations of the angular deviation and yaw due to various disturbing forces and initial conditions, and based upon the six independent parameters V_0, V_1, f, γ, r^2 and β, were made at 28 different points during burning, corresponding to velocities of 150, 200, 250, ..., 1500 ft./sec.

Curves of the types described in § 3·31 have been drawn from the values calculated, and are given in figures 6 to 15, the points corresponding to each 200 ft./sec. of velocity being marked.

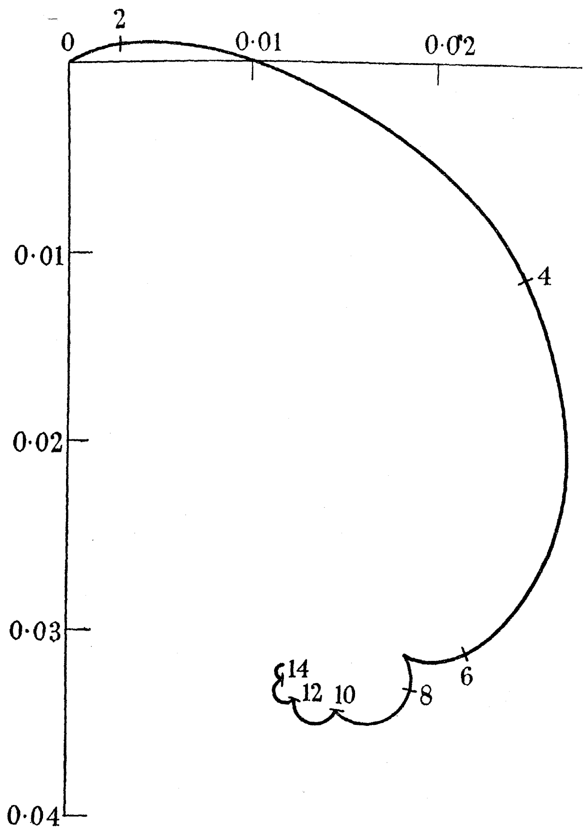

FIGURE 14. Angular deviation due to a displacement of the principal longitudinal axis of inertia. Displacement $= \alpha_C \times 10^{-3}$ radian downwards (forward end).

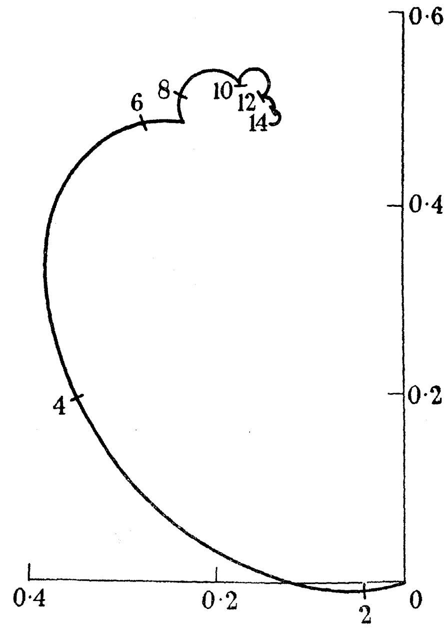

FIGURE 15. Angular deviation due to a malalinement of the thrust.

Figure 14. *Displacement of the longitudinal principal axis of inertia.* The figure shows the angular deviation due to a displacement $\alpha_C = 10^{-3}$ radian downwards in the vertical plane at launch. Perfect launch is assumed. No curve for the yaw is given since a far greater number of points than the 28 used would be required to record accurately the subsidiary high-frequency oscillations which are imposed upon the two normal modes of precession.

Figure 15. *Malalined thrust.* The figure shows the angular deviation due to a thrust malalinement $\alpha_R = 10^{-3}$ radian downwards in the vertical plane at launch when $ml = A\gamma$. When this relation does not hold, the co-ordinates should be multiplied by $ml/A\gamma$. No curve for the yaw is given for the same reason as mentioned above.

APPENDIX C. COMPARATIVE LIST OF SYMBOLS USED BY VARIOUS AUTHORS

Kelley, McShane & Reno 1949?	Nielsen & Synge (1946)	Fowler *et al.* (1920) *Text-book of A.A. gunnery* (1925)	Rankin (present paper)	remarks
ρ	ρ	ρ	ρ	—
d	$2a$	$2r$	$2a$	—
$4K_N$	f_1	f_R+f_L	$k/\rho a^2$	—
$8K_F$	f_2	f_K	$k_M/\rho a^3$	—
$16K_{XF}$	g_1	—	$-k'_M/\rho a^4$	—
$8K_S$	g_2	—	$-k'/\rho a^3$	—
$16K_T$	f'_1	f_J	$-k_M d_3/\rho a^4$	—
$8K_M$	f'_2	f_M	$-kd_1/\rho a^3$	—
$16K_H$	g'_1	f_H	$k'd_2/\rho a^4$	—
$32K_{XT}$	g'_2	—	$k'_M d_4/\rho a^5$	—
$4K_{DA}$	f_3	f_R	$R_A/\rho a^2$	—
$16K_A$	g_3	f_I	$\Gamma_A/\rho a^3$	—
m	m	$\mathbf{m}$	m	—
A	C	A	C	—
B	A	B	$A=B$	—
u	w	v	V	—
ω_1	ω_3	N	r	—
$A\omega_1/B$	$C'\omega_3$	Ω	$Cr/A=2\beta r$	—
ξ	ξ	—	$-V\Xi$	see note 1
$\lvert\xi\rvert$	$\lvert\xi\rvert$	$v \sin \delta$	$V\delta$	—
η	η	—	$i(d\zeta/dt)$	see note 1
$\lvert\eta\rvert$	$\lvert\eta\rvert$	w	ω	—
Θ	—	θ	α	—
ζ	—	—	$-\Xi$	see note 1
$\rho d^2 s_1$	—	—	$2m\alpha_2$	see note 2
$\rho d^2 s_2$	—	—	$2m(\alpha_2+\alpha_1+\varpi/2\beta)$	see note 2
$\rho d^2 s_3$	—	—	$2m(\alpha_2-\alpha_1-\varpi/2\beta)$	see note 2
$\rho^2 d^4 s_2 s_3$	—	—	$m^2 S_1/\beta_1^2 V^2$	see note 2

Note 1. Owing to the different systems of axes used the quantities ξ, η, and ζ in columns 1 and 2 are not strictly equal to the quantities $-V\Xi$, $i(d\zeta/dt)$ and Ξ in column 4. They differ, however, only in factors of unit modulus.

Note 2. This equivalence is approximate.

APPENDIX D. INDEX OF SYMBOLS

The index gives the number of the section, subsection or formula where the symbol is defined. Symbols occurring in only one subsection are not listed in every case. In § 3·24 a list of those symbols which can carry the suffices 00, 0 and 1 in order to denote values at ignition, launch and burnt is given. Names of points in figure 3 are not given here. A dagger † denotes that the suffix P stands for any one of C, G, L, M, N, R.

Italic letters

A	2·4, 3·221	B	2·2 (body); 2·4, 3·221 (M.I.)
$A(u)$	(6·4·4, 5)	$B(u)$	(6·4·4, 5)
$A^*(u)$	(6·82·1)	$B^*(u)$	(6·82·1)
$A_1(u)$	(6·4·6)	$\mathbf{b}$	2·4
$\mathbf{a}$	2·4	b_p	9·41
a, a_M	3·221	$b(u, v)$	(6·4·21)
a_F	3·53	$\dagger b_P(\nu)$	3·63, 6·10
a_e	9·41		
a_p, a'_p	9·41	C	2·4, 3·221
$a(u, v)$	(6·4·20)	Ci x	(6·82·2)
$\dagger a_P(\nu)$	3·63, 6·10	$C(u)$	(6·4·2)

$C(u, v)$	(6·4·3)
$\mathbf{c}$	2·4
c	(9·2·4)
D	(3·3·13)
$D(u)$	(6·4·4)
$D^*(u)$	(6·82·1)
d_1	(3·513·3)
d_2	(3·514·3)
d_3	(3·515·3)
d_4	(3·516·3)
E	(4·2·12)
E_1, E_2	(6·4·22)
$E(u, v)$	(6·4·10)
$E^*(u, v)$	(6·4·11)
$\mathscr{E}(u, v)$	(6·4·1)
e	2·71828...
e_1, e_2	(6·82·3)
$e(u)$	(6·82·2)
$\mathbf{F}$	(3·51·1)
F	(4·2·13)
F_1	(4·2·14)
F_1, F_2	3·511 (components of $\mathbf{F}$)
F^*	(3·511·1)
$F(u, v)$	(6·4·13)
f	3·23
f_1, f_2	(3·511·1), (3·517·11)
f_Q, f_R	(4·2·1)
$\mathbf{G}$	(3·51·1)
G^*	(3·511·2)
G_1, G_2	3·511 (components of $\mathbf{G}$)
G_0, G_1	(6·3·4)
$G_{01}, G_{02}, G_{11}, G_{12}$	(6·3·6)
$\mathbf{G}_R$	(2·5·10), 3·214
$G(s)$	(4·2·17)
$G^*(s)$	(8·2·2)
$G_1(s), G_2(s)$	(4·2·18, 19)
$G_3(s)$	(8·3·5)
$G_4(s)$	(8·41·7)
$G(u, v)$	(6·4·12)
$\mathbf{g}, g$	3·5
g_1, g_2	(3·511·2), (3·517·11)
$g_1^*(s), g_2^*(s)$	(8·3·3)
$g(u, v)$	(8·2·4)
H	see eta
$H(u, v)$	(6·4·14)
$\mathbf{h}$	2·4
$\mathbf{h}_G$	(2·4·3), 3·23
h_0, h_1	6·81
h_e	(6·85·4)
i	$\surd(-1)$
$j(t_0, t)$	(7·32·8)
K_1, K_2, K_3	constants
$K(u, v)$	(6·84·2)
$K_1(u, s), K_2(u, s)$	8·41
$\mathbf{k}$	2·5, 3·214
k	(3·513·3)
k'	(3·514·3)
k''	(3·518·3)

k_M	(3·515·3)
k'_M	(3·516·3)
k_e	2·5, 3·223
$k_1(u, s), k_2(u, s)$	8·41
$\mathbf{L}$	2·4
$\mathbf{L}_1, L_1$	(3·51·2), 3·513
$\mathbf{L}_2, L_2$	(3·51·2), 3·514
$\mathbf{L}_3, L_3$	(3·51·2), 3·515
$\mathbf{L}_4, L_4$	(3·51·2), 3·516
$\mathbf{L}'_1, L'_1$	3·52
$\mathbf{L}'_3, L'_3$	3·52
L''_1	3·518
$\mathbf{L}_A$	3·5
$\mathbf{L}_I$	3·51
$\mathbf{L}_S$	3·5
$\mathbf{L}_e$	(3·5·2)
l	2·5, 3·22·1
l_c	9·2
$\mathbf{M}$	2·4
$\mathbf{M}_1, M_1$	(3·51·3), 3·513
$\mathbf{M}_2, M_2$	(3·51·3), 3·514
$\mathbf{M}_3, M_3$	(3·51·3), 3·515
$\mathbf{M}_4, M_4$	(3·51·3), 3·516
$\mathbf{M}'_1, M'_1$	3·52
$\mathbf{M}'_3, M'_3$	3·52
$\mathbf{M}_A$	3·5
$\mathbf{M}_I$	3·51
$\mathbf{M}_s$	3·5
$\mathbf{M}_e$	(3·5·2)
m	2·3, 3·221
N	10·31 (nozzles)
N_c	9·2
$\mathbf{n}$	(2·5·3), 3·214
n^2, n	(4·2·2)
n_p	10·31
$\mathbf{P}$	2·4
P	3·53, 9·6
$P(s)$	(4·2·15)
$P(u, v)$	(6·4·23)
$\mathbf{p}, p$	3·5
p	(5·1·3)
p_1, p_2	(5·1·5)
Q	(2·3·2), 3·223
$Q(u, v)$	(6·4·24)
$Q_{11}, Q_{12}, Q_{21}, Q_{22}(u, s)$	(8·41·15)
q	3·222
q_1	2·5, 3·223
q_2	(2·4·10), 3·223
$q_{11}, q_{12}, q_{21}, q_{22}(u, s)$	(8·41·12)
$\mathbf{R}$	2·2 (radius); 3·51, 3·512 (drag)
R_1, R_2	(6·101·1)
R_A	(3·51·2)
$\mathbf{R}_G$	2·5
$\mathbf{R}_M$	(2·4·10), 3·214
$\mathbf{R}_N$	(2·3·7)
R_c	9·2
$R(s)$	(5·2·1)
$R_1(s)$	(5·2·4)
$R_2(s)$	(5·2·5)

$R_{\mathrm{I}}(s)$, $R_{\mathrm{II}}(s)$	8·41
r	3·23
$\mathbf{r}$, $\mathbf{r}_G$, $\mathbf{r}_H$	2·2
r'	7·32
r_c	9·2
S	(8·41·8)
S_0	2·2
S_1, S_2	(8·42·4, 5)
S_3	(8·42·7)
Si x	(6·82·2)
$S(u)$	(6·4·2)
$S(u, v)$	(6·4·3)
s	3·23
s_e	(6·85·4)
si x	(6·82·2)
$T(s)$	(4·2·20)
$T_1(s)$, $T_2(s)$	(4·2·21, 22)
$T_3(s)$	(5·2·2)
t	2·2, 3·23
$\mathbf{U}$	2·2
U	8·5 (aircraft speed)
u	used for s under sign of integration
$\mathbf{V}$, V	2·2, 3·23
V^*	(3·511·3)
V_1, V_2	3·511
$\mathbf{V}_H$	2·2
V_W	(3·52·2)
V_e	(6·85·3)
V_u, V_v	used for V under sign of integration
$\mathbf{v}$	2·2
v	used for s under sign of integration
v_0, v_1	(6·3·9)
v_N	2·2
$\mathbf{W}$, W	(2·3·8), 3·22·3
W'	9·41
$\mathbf{w}$, w	3·215
w_1	(3·3·14)
w_{01}, w_{02}, w_{11}, w_{12}	(6·3·5)
w_F, w_L	5·6
$x_1(\mu)$, $x_2(\mu)$, $x_3(\mu)$	(5·7·6, 7, 8)
$y_1(\mu)$, $y_2(\mu)$, $y_3(\mu)$	(5·7·9, 10, 11)
Z	see zeta
$Z(u)$, $Z_1(u)$	(6·4·7)
$z_1(\mu)$, $z_2(\mu)$, $z_3(\mu)$	(5·7·12 to 5·7·15)

Greek letters

α	3·211
α_1, α_2	(4·2·4)
α_K	3·214
†α_P	3·214
β	(4·2·6)
β_1, β_2	(4·2·5)
$\mathbf{\Gamma}$	3·31, 3·512
Γ_A	(3·512·2)
Γ_F	(3·53·4)
$\mathbf{\Gamma}_R$	3·53
γ	(5·1·1)
Δ_F	3·53
Δ_e	9·41
Δ_p	9·41
Δ_ρ	(3·53·1)
δ	3·213
ϵ	6·81
Z	(3·3·1)
$\mathrm{Z}^{(1)}$, $\mathrm{Z}^{(2)}$	(6·4·25)
Z_N, Z_{NT}	6·9
Z_R, Z_{RT}	6·9
†Z_P	6·9, 6·10
Z_T	(6·7·1)
Z_g	4·4, (6·7·1)
ζ	(3·3·2)
ζ_{01}	(3·7·1)
$d\zeta^{(1)}/dt$, $d\zeta^{(2)}/dt$	(6·4·27)
†$d\zeta_p/dt$	6·9, 6·10
$d\zeta_g/dt$	6·7
H, $\mathrm{H}(s)$	(4·2·23)
H_{01}	(4·4·9)
$\mathrm{H}^{(1)}$, $\mathrm{H}^{(2)}$	8·41
η	(3·3·15)
Θ	3·212
θ	3·211, 3·213
κ	(4·2·7)
κ_A, κ_C	(9·2·2, 3)
Λ, $\Lambda(s)$	(4·2·16)
λ	(4·2·8)
λ_1, λ_2	(6·3·3)
μ_1, μ_2, μ_3	(4·2·9, 10, 11)
ν, ν^2	(4·2·2) (aerodynamic parameter)
ν	(9·3·6) (spin parameter)
†ν_P	8·41
Ξ	(3·3·3)
$\Xi^{(1)}$, $\Xi^{(2)}$	(6·4·26), (8·41·5)
$\Xi_{\mathrm{I}}^{(2)}$, $\Xi_{\mathrm{II}}^{(2)}$	8·41
Ξ_N, Ξ_R	6·9
†Ξ_P	6·9, 6·10
Ξ_g	6·7
ξ	3·215 (wind); (7·2·3) (spin)
ξ_1, ξ_2	3·215
π	3·14159...
ρ	2·2 (body density); App. C (air density); 3·53 (radius)
ρ_c	9·2

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